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1 **THE GENERALISED SINGULAR PERTURBATION**
2 **APPROXIMATION FOR BOUNDED REAL AND POSITIVE**
3 **REAL CONTROL SYSTEMS**

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ABSTRACT. The generalised singular perturbation approximation (GSPA) is considered as a model reduction scheme for bounded real and positive real linear control systems. The GSPA is a state-space approach to truncation with the defining property that the transfer function of the approximation interpolates the original transfer function at a prescribed point in the closed right half complex plane. Both familiar balanced truncation and singular perturbation approximation are known to be special cases of the GSPA, interpolating at infinity and at zero, respectively. Suitably modified, we show that the GSPA preserves classical dissipativity properties of the truncations, and existing *a priori* error bounds for these balanced truncation schemes are satisfied as well.

4 **1. Introduction.** Model reduction of finite-dimensional, continuous-time, linear
5 control systems of the form

$$\left. \begin{aligned} \dot{x} &= Ax + Bu, & x(0) &= x^0, \\ y &= Cx + Du, \end{aligned} \right\} \quad (1.1)$$

6 by the generalised singular perturbation approximation (GSPA) is considered. Here,
7 as usual, u , x and y denote the input, state and output, respectively, and A , B ,
8 C and D are appropriately sized matrices. Model reduction in this context refers
9 to approximating the input-output relationship $u \mapsto y$ in (1.1) by a simpler one,
10 which is ideally both qualitatively and quantitatively close to the original. Model
11 reduction is important for both simulation and controller design [39]. There are a
12 multitude of different approaches to model reduction in the literature, see [13] and
13 in particular [13, Fig. 2.1], including, for example, state-space methods, polynomial
14 and rational interpolation and error minimisation methods to name but a few. The
15 GSPA is in the spirit of the classic control theoretic model reduction scheme called
16 (Lyapunov) balanced truncation, proposed in [31], and its close relation, the singular
17 perturbation approximation, first considered in the context of model reduction of
18 linear control systems in [11, 12].

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1 Lyapunov balanced realisations of stable systems are computed by finding a state-
 2 space similarity transform under which the solutions P and Q of the controller and
 3 observer Lyapunov equations, respectively,

$$AP + PA^* + BB^* = 0 \quad \text{and} \quad A^*Q + QA + C^*C = 0,$$

4 are equal. States in (1.1) are omitted in a reduced order model, the so-called bal-
 5 anced truncation, according to the relative size of the square roots of the eigenvalues
 6 of the product PQ (which are similarity invariants), which are in fact equal to the
 7 singular values of the Hankel operator associated with (1.1). Lyapunov balanced
 8 truncations retain stability and minimality of the original model — properties es-
 9 tablished in [40] — and another appealing property is the *a priori* error bound
 10

$$\|\mathbf{G} - \mathbf{G}_r\|_{H^\infty} \leq 2 \sum_{j=r+1}^n \sigma_j, \quad (1.2)$$

11 between the transfer function \mathbf{G} and its reduced order approximation \mathbf{G}_r . Here
 12 σ_j denote the distinct Hankel singular values, and the summation on the right
 13 hand side of (1.2) contains the singular values *omitted* from the reduced-order sys-
 14 tem. The error bound (1.2) was derived independently in [10] and [15]. The upper
 15 bound (1.2) is known to be achieved (that is, equality holds in (1.2)) for certain
 16 single-input single-output (SISO) systems, see [29], and a lower bound in the multi-
 17 input multi-output (MIMO) case has recently been derived in [38]. For more infor-
 18 mation on balanced truncation, the reader is referred to the survey paper [18] or the
 19 textbooks [3, 13, 17, 36]. The popularity of balanced realisations and balanced trun-
 20 cation has led to numerous further developments, some of which we discuss further
 21 below, as well as, for example, to infinite-dimensional systems: [8, 14, 16, 24, 35, 49].

22 In the frequency domain, balanced truncation for rational functions is a model
 23 reduction scheme which yields a rational approximation with the property that it
 24 interpolates the original function at infinity. Roughly, by applying the same method
 25 to a rational function now with argument $1/s$ instead of s , another reduced order
 26 rational transfer function is obtained, which now interpolates the original at zero.
 27 Interpolating at zero is a frequency domain property of the so-called singular per-
 28 turbation approximation (SPA), in particular meaning that the steady-state gains
 29 are equal. From a dynamical systems perspective, singular perturbation approxi-
 30 mation decomposes the state variables into those with “fast” and “slow” dynamics,
 31 and assumes that the “fast” variables are at equilibrium, meaning that differen-
 32 tial equations simplify to algebraic equations. For linear systems these algebraic
 33 equations are easily solvable, which leads to a model with fewer differential equa-
 34 tions, and hence fewer states. The mapping s to $1/s$ mentioned above is called the
 35 reciprocal transformation and provides a relationship between SPA and balanced
 36 truncation. This relationship was exploited in [30] to show that the singular per-
 37 turbation approximation of a balanced, minimal, linear system admits the same
 38 H^∞ error bound (1.2), as well as retaining minimality and stability of the original.
 39 To the best of our knowledge, the provenance of the reciprocal transformation in
 40 systems and control theory is unclear, and it now forms part of the subjects’ “folk-
 41 lore”. It appears in numerous areas, for instance, when working with the technical
 42 difficulties which arise in infinite dimensional systems see, for example, [44, Section
 43 12.4] and [6, 7].

1 The generalised singular perturbation approximation (GSPA) is a generalisation of
 2 both balanced truncation and singular perturbation approximation as it is a state-
 3 space truncation scheme with the property that the approximate transfer function
 4 interpolates the original at a prescribed point in the closed right half complex plane.
 5 The GSPA was proposed in a control theoretic context in [11], and was the subject of
 6 a number of papers around that time, see [1, 30, 27, 32]. Both balanced truncation
 7 and the SPA are special cases of the GSPA.

8 Here we demonstrate that when suitably adapted, the GSPA provides a dissipativity
 9 preserving model reduction scheme with error bounds and the additional interpo-
 10 lation property. To motivate our study we note that a disadvantage of balanced
 11 truncation or SPA is that any dissipativity property of the original system need not
 12 be retained in the truncation. Dissipativity (or passivity) theory as commonly used
 13 in systems and control theory dates back to the seminal work of [47, 48], where
 14 the notions of supply rate and storage function were introduced and which capture
 15 (and generalise) the notion of a system storing and dissipating energy over time.
 16 Dissipative systems are central to control, in part owing to a plethora of natural and
 17 important examples such as RLC circuits and mass-spring-damper systems. Much
 18 attention has been devoted to the situation when the supply rate is *quadratic*, as
 19 multiple notions of energy are quadratic in state variables, such as kinetic energy.
 20 Two classical notions of quadratic dissipative systems which first arose in circuit
 21 theory go by the names of impedance passive and scattering passive, also known as
 22 passive and contractive, or bounded real and positive real, respectively, the latter
 23 term being introduced in [4]. Two famous results, sometimes called the Bounded
 24 Real Lemma and Positive Real Lemma, provide a complete state-space characteri-
 25 sation of these two notions of dissipativity, respectively, see, for example, [2]. The
 26 latter is also known as the Kalman-Yakubovich-Popov (KYP) Lemma in recognition
 27 of its original contributors. We refer the reader to [25] or [41] and the references
 28 therein for more background on the KYP Lemma.

29 In response, balanced truncation has been extended to bounded real and positive
 30 real systems in [9] and [37], respectively, and to the infinite-dimensional case in [22].
 31 Here the truncations *do* retain the respective dissipativity property and error bounds
 32 have also been established see, for example, [18] and [23]. We note that there is a
 33 false bound in [5], see [21]. By using the reciprocal transformation, it was shown
 34 in [33] that when the SPA is defined in terms of a dissipative balanced realisation,
 35 then the reduced order system inherits dissipativity from the original system, and
 36 satisfies corresponding error bounds. There have been other variations in dissi-
 37 pativity preserving model reduction schemes, including to descriptor systems [42],
 38 and certain classes of finite-dimensional behavioral systems [23]. To summarise, the
 39 bounded real and positive real GSPA generalises the results of [9], [37] and [33] and
 40 provides a truncation scheme which retains the relevant dissipativity property, error
 41 bounds, and interpolation at a prescribed point.

42 The manuscript is organised as follows. After recording notation and terminology,
 43 Section 2 recalls model reduction by generalised singular perturbation approxima-
 44 tion. Our main results are contained in Sections 3 and 4, namely, bounded real and
 45 positive real preserving generalised singular perturbation approximation. Examples
 46 are contained in Section 5. In an attempt to streamline the presentation, the proofs
 47 of our main results appear in Section 6 and the Appendix.

Notation: Most mathematical notation we use is standard, or defined when introduced. The set of positive integers is denoted by \mathbb{N} , whilst \mathbb{R} and \mathbb{C} denote the fields of real and complex numbers, respectively. For $k \in \mathbb{N}$, $\underline{k} := \{1, 2, \dots, k\}$ and for $\xi \in \mathbb{C}$, $\operatorname{Re}(\xi)$, $\operatorname{Im}(\xi)$, $\bar{\xi}$ and $|\xi|$ denote its real part, imaginary part, complex conjugate and modulus, respectively. We let \mathbb{C}_0 denote the set of all complex numbers with positive real part. For $n \in \mathbb{N}$, \mathbb{R}^n and \mathbb{C}^n denote the familiar real and complex n -dimensional Hilbert spaces, respectively, both equipped with the inner product $\langle \cdot, \cdot \rangle$ which induces the usual 2-norm $\|\cdot\|_2$. For $m \in \mathbb{N}$, let $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$ denote the normed linear spaces of $n \times m$ matrices with real and complex entries, respectively, both equipped with the operator norm, also denoted $\|\cdot\|_2$, induced by the $\|\cdot\|_2$ norm on \mathbb{R}^n or \mathbb{C}^n . The superscript $*$ denotes the complex-conjugate transpose (and, importantly, the adjoint with respect to the above inner product). For $M, N \in \mathbb{C}^{n \times n}$, $\sigma(M)$ denotes the spectrum of M and we write $M \geq N$ or $N \leq M$ if $M - N$ is positive semi-definite, and $M > N$ or $N < M$ if the difference $M - N$ is positive definite. It is well-known that, as \mathbb{C}^n is a complex Hilbert space, if $M \succeq 0$, then $M = M^*$. For $m, p \in \mathbb{N}$, the space of analytic functions $\mathbb{C}_0 \rightarrow \mathbb{C}^{p \times m}$ is denoted by $H(\mathbb{C}_0, \mathbb{C}^{p \times m})$. The subset of functions which are additionally bounded with respect to the norm

$$\|\mathbf{G}\|_{H^\infty} = \sup_{s \in \mathbb{C}_0} \|\mathbf{G}(s)\|_2,$$

is denoted by $H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times m})$.

2. The generalised singular perturbation approximation. We gather elementary and notational preliminaries before recalling the generalised singular perturbation approximation and describing some properties.

We consider the linear control system (1.1) where, as usual, u , x and y denote the input, state and output and

$$(A, B, C, D) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{p \times n} \times \mathbb{C}^{p \times m},$$

for some $m, n, p \in \mathbb{N}^1$. In practice, the quadruple (A, B, C, D) is real-valued and in many situations, the matrix D does not play a role. As such, we use the triple (A, B, C) when the choice of D , which need not be zero, is unimportant.

The triple (A, B, C) is said to be stable if A is Hurwitz, that is, every eigenvalue of A has negative real part. The dimension of the triple (A, B, C) is equal to the dimension of its A term, and the triple is minimal if the pair (A, B) is controllable and the pair (C, A) is observable, see [43, Theorem 27, p.286].

Naturally, associated to the quadruple (A, B, C, D) is the linear system (1.1). The transfer function of the linear system (1.1) or quadruple (A, B, C, D) is the rational function

$$s \mapsto \mathbf{G}(s) := D + C(sI - A)^{-1}B, \quad (2.1)$$

which is certainly defined for all complex s with $\operatorname{Re} s > \alpha(A)$, the spectral abscissa of A . Conversely, given a proper rational function \mathbf{G} defined on a right-half complex plane, a quadruple (A, B, C, D) is called a realisation of \mathbf{G} if (2.1) holds on that half-plane. Realisations are never unique. The McMillan degree of a proper rational

¹The material which follows holds if we assume that $A : \mathcal{X} \rightarrow \mathcal{X}$, $B : \mathcal{U} \rightarrow \mathcal{X}$, $C : \mathcal{X} \rightarrow \mathcal{Y}$ and $D : \mathcal{U} \rightarrow \mathcal{Y}$ are bounded linear operators between finite-dimensional complex Hilbert spaces \mathcal{U} , \mathcal{X} and \mathcal{Y} which, of course, is equivalent to our formulation once bases are chosen for \mathcal{U} , \mathcal{X} and \mathcal{Y} .

1 transfer function is the dimension of a minimal state-space realisation, see [43,
2 Remark 6.7.4, p.299].

3 Recall that the stable triple (A, B, C) is called (internally or Lyapunov) balanced
4 if there exists a Σ such that

$$A\Sigma + \Sigma A^* + BB^* = 0 \quad \text{and} \quad A^*\Sigma + \Sigma A + C^*C = 0. \quad (2.2)$$

5 If Σ satisfies (2.2), then necessarily Σ equals both the controllability and observ-
6 ability Gramians of the linear system specified by (A, B, C) , that is,

$$\Sigma = \int_{\mathbb{R}_+} e^{At} BB^* e^{A^*t} dt = \int_{\mathbb{R}_+} e^{A^*t} C^* C e^{At} dt,$$

7 (hence the terminology balanced) and is consequently self-adjoint and positive semi-
8 definite. It is well-known that it is always possible to construct a balanced realisation
9 from a given one via a state-space similarity transformation [3, Lemma 7.3, p.210].
10 The triple (A, B, C) is minimal if, and only if, Σ is positive definite. The eigenvalues
11 of Σ are precisely the singular values of the Hankel operator corresponding to the
12 triple (A, B, C) . We shall let $(\sigma_j)_{j=1}^n$ denote the n *distinct* Hankel singular values
13 of (A, B, C) , which we shall assume throughout the paper are simple (that is, each
14 has algebraic and geometric multiplicity equal to one). As singular values, the σ_j
15 are ordered so that

$$\sigma_1 > \sigma_2 > \cdots \geq 0. \quad (2.3)$$

16 In practical applications, a basis of the state-space is chosen so that Σ is a diagonal
17 matrix, with the terms σ_j on the diagonal.

18 Singular perturbation approximations are defined in terms of conformal partitions
19 of (A, B, C) , denoted by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (C_1 \quad C_2), \quad (2.4)$$

20 where $A_{11} \in \mathbb{R}^{r \times r}$, $B_1 \in \mathbb{R}^{r \times m}$, $C_1 \in \mathbb{R}^{p \times r}$ and so on, for some $r \in n - 1$. Of course,
21 the partitions in (2.4) depend on both the realisation and r , which are degrees of
22 freedom.

23 **Definition 2.1.** Given the quadruple (A, B, C, D) , partitioned according to (2.4)
24 for some $r \in n - 1$ and $\xi \in \mathbb{C}$, $\text{Re}(\xi) \geq 0$ assume that $\xi \notin \sigma(A_{22})$. The quadruple
25 $(A_\xi, B_\xi, C_\xi, D_\xi)$ given by

$$\left. \begin{aligned} A_\xi &:= A_{11} + A_{12}(\xi I - A_{22})^{-1} A_{21}, & B_\xi &:= B_1 + A_{12}(\xi I - A_{22})^{-1} B_2, \\ C_\xi &:= C_1 + C_2(\xi I - A_{22})^{-1} A_{21}, & D_\xi &:= D + C_2(\xi I - A_{22})^{-1} B_2, \end{aligned} \right\} \quad (2.5)$$

26 is called the generalised singular perturbation approximation of (1.1).

27 *Remark 2.2.* Throughout this remark, we assume that $\xi \in \mathbb{C}$, $\text{Re}(\xi) \geq 0$.

28 (a) The generalised singular perturbation approximation may be defined for any
29 realisation (A, B, C) and choice of partition in (2.4). In this section we shall assume
30 that (A, B, C) is stable and balanced and a partition in (2.4) is chosen with respect
31 to two unions of eigenspaces of Σ corresponding to *distinct* eigenvalues. With
32 respect to such a partition, Σ has the block form

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (C_1 \quad C_2). \quad (2.6)$$

1 In light of the ordering (2.3), Σ_1 and Σ_2 contain the larger and smaller eigenvalues
2 of Σ , respectively.

3 (b) Given the stable, minimal, balanced quadruple (A, B, C, D) with transfer func-
4 tion \mathbf{G} , let \mathbf{G}_r^ξ denote the transfer function of the generalised singular perturbation
5 approximation. The motivation and defining property of the generalised singular
6 perturbation approximation is that

$$\mathbf{G}_r^\xi(\xi) = \mathbf{G}(\xi), \quad (2.7)$$

7 that is, the transfer function interpolates the original at ξ , see [30, Lemma 2.4]. Of
8 course, a downside of the GSPA for practical applications is that $(A_\xi, B_\xi, C_\xi, D_\xi)$
9 will in general be complex when $\text{Im } \xi \neq 0$, even if (A, B, C, D) is real.

10 (c) If the realisation (A, B, C) is stable, minimal, balanced and decomposed as
11 in (2.6), then by [40, Theorem 3.2] both A_{11} and A_{22} are Hurwitz. Consequently,
12 the generalised singular perturbation approximation is well-defined. Furthermore,
13 in the limit $\xi \rightarrow \infty$, we obtain from (2.5)

$$A_\infty := A_{11}, \quad B_\infty := B_1, \quad C_\infty := C_1, \quad D_\infty := D,$$

14 and the linear system specified by the quadruple $(A_\infty, B_\infty, C_\infty, D_\infty)$ is called the
15 balanced truncation of (1.1). The case $\xi = 0$ in (2.5) leads to

$$\begin{aligned} A_0 &:= A_{11} - A_{12}A_{22}^{-1}A_{21}, & B_0 &:= B_1 - A_{12}A_{22}^{-1}B_2, \\ C_0 &:= C_1 - C_2A_{22}^{-1}A_{21}, & D_0 &:= D - C_2A_{22}^{-1}B_2, \end{aligned}$$

16 and the linear system specified by the quadruple (A_0, B_0, C_0, D_0) is called the sin-
17 gular perturbation approximation of (1.1). We see that the balanced truncation
18 and singular perturbation approximation are special cases of the generalised sin-
19 gular perturbation approximation, hence the terminology. In state-space terms,
20 the GSPA assumes that x in (1.1) is partitioned into x_1 and x_2 and

$$\dot{x}_2(t) = \xi x_2(t). \quad (2.8)$$

21 By substituting (2.8) into (1.1) and eliminating x_2 , the linear system specified
22 by $(A_\xi, B_\xi, C_\xi, D_\xi)$ is obtained (with state x_1). The assumption (2.8) highlights
23 the input-output motivation of the GSPA, at least for stable systems. Indeed,
24 $\dot{x}_2(t) = \xi x_2(t)$ and $\text{Re}(\xi) \geq 0$ implies that $\|x_2(t)\|$ does not decrease as $t \rightarrow \infty$.
25 Under the assumption that A is Hurwitz, we would of course expect $\|x_2(t)\| \rightarrow 0$
26 as $t \rightarrow \infty$ in the absence of control, that is, when $u = 0$. \diamond

27 We recall two results which shall play a key role in constructing the dissipativity
28 preserving GSPA in Sections 3 and 4.

29 **Theorem 2.3.** *Given $\xi \in \mathbb{C}$ with $\text{Re}(\xi) \geq 0$ and stable, minimal, balanced quadruple*
30 *(A, B, C, D) , assume that the Hankel singular values are simple. Then $(A_\xi, B_\xi, C_\xi, D_\xi)$,*
31 *the generalised singular perturbation approximation of order $r \in \underline{n-1}$, is well-*
32 *defined and the following statements hold.*

- 33 (i) A_ξ is Hurwitz and (A_ξ, B_ξ, C_ξ) is minimal.
34 (ii) If $\xi \in i\mathbb{R}$, then (A_ξ, B_ξ, C_ξ) is balanced.

35 Statement (i) of Theorem 2.3 appears in the special case that $\xi \in \mathbb{R}$, $\xi > 0$ in [27,
36 Theorem 5.4], but does not appear in [30]. It is claimed in [27, Remark 5.5] that
37 statement (i) extends to all $\xi \in \mathbb{C}_0$, but no proof is given there. For completeness,
38 we have provided a proof in the Appendix. Statement (ii) is novel.

Theorem 2.4. *Let $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times m})$ be rational with simple Hankel singular values $(\sigma_j)_{j=1}^n$, ordered as in (2.3), let $r \in \overline{n-1}$ and $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$. Then there exists a rational $\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times m})$ of McMillan degree r such that the interpolation property (2.7) holds and*

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^n \sigma_j. \quad (2.9)$$

The proof of Theorem 2.4 is constructive — a transfer function \mathbf{G}_r^ξ which satisfies (2.7) and (2.9) is realised by the generalised singular perturbation approximation of a stable, minimal, balanced realisation of \mathbf{G} . The error bound (2.9) has been established when $\xi = 0$ or $\xi = \infty$ as these correspond to the singular perturbation approximation and balanced truncation, respectively, as well as when $\xi \in i\mathbb{R}$ (see [30, Theorem 3.4]) and when $\xi \in \mathbb{R}$, $\xi > 0$ (see [27, Theorem 5.4]). Again, it is claimed in [27, Remark 5.5] that the error bound (2.9) holds for all $\operatorname{Re}(\xi) \geq 0$, but no proof is given. Again for completeness, a proof is provided in the Appendix.

3. Bounded real generalised singular perturbation approximation. In this section we define the bounded real GSPA of a quadruple with bounded real transfer function, and show that it gives rise to a bounded real reduced order system, with properties including the point interpolation (2.7) and error bounds. Recall that $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times m})$ is said to be bounded real if $\|\mathbf{G}\|_{H^\infty} \leq 1$, and strictly bounded real if $\|\mathbf{G}\|_{H^\infty} < 1$. Bounded realness is the frequency domain name of the property called scattering passive or contractive in the time-domain. From many possible references the reader is referred to, for example, [45, 46]. The term ‘real’ in bounded real refers to the sometimes-made assumption that \mathbf{G} is real on the real axis. It is true that many physically motivated systems enjoy such a property, but we do not enforce it because there is no mathematical need to. Although we acknowledge that the terminology ‘bounded’ or ‘contractive’ would suffice, in keeping with existing literature we persevere with the term ‘bounded real’.

Bounded real balanced truncation, proposed in [37], and bounded real singular perturbation approximation, proposed in [33], are morally similar to the (Lyapunov) balanced versions. However, instead of balancing the solutions of two Lyapunov equations, for the bounded real model reduction schemes certain solutions of the so-called primal and dual Bounded Real Lur’e (or Algebraic Riccati) equations are balanced. The existence of these solutions is ensured by the Bounded Real Lemma. There are numerous treatments of bounded real balanced truncation in the literature, examples in addition to [37] and [33] include [3, 18, 19, 22, 23]. For brevity, here we describe only the aspects required to define the bounded real GSPA.

For which purpose, recall that if the stable, minimal quadruple (A, B, C, D) is bounded real, then there exist P_m and P_M , positive definite solutions of the Bounded Real Lur’e equations

$$\left. \begin{aligned} A^*Z + ZA + C^*C &= -K^*K, \\ ZB + C^*D &= -K^*W, \\ I - D^*D &= W^*W, \end{aligned} \right\} \quad (3.1)$$

(with variable Z), for some $K \in \mathbb{C}^{m \times n}$ and $W \in \mathbb{C}^{m \times m}$, which are extremal in the sense that any other positive semi-definite solution P of (3.1) satisfies $P_m \leq P \leq$

1 P_M . It is straightforward to show that P_M^{-1} is also equal to the minimal solution
 2 (in the previous sense) of the dual Bounded Real Lur'e equations

$$\left. \begin{aligned} AZ + ZA^* + BB^* &= -LL^*, \\ ZC^* + BD^* &= -LX^*, \\ I - DD^* &= XX^*, \end{aligned} \right\} \quad (3.2)$$

3 (also with variable Z) for some $L \in \mathbb{C}^{n \times p}$ and $X \in \mathbb{C}^{p \times p}$. We say that the realisation
 4 (A, B, C, D) is bounded real balanced if

$$P_m = P_M^{-1} =: \Sigma.$$

5 In particular, when (A, B, C, D) is bounded real balanced, then Σ is a solution of
 6 both (3.1) and (3.2). The bounded real singular values, denoted $(\sigma_k)_{k=1}^n$, are the
 7 nonnegative square roots of the eigenvalues of $P_m P_M^{-1}$, and so the eigenvalues of Σ
 8 in a bounded real balanced realisation. We note that they are called characteristic
 9 values by some authors, such as in [42]; see [23, Remark 3.6].

10 **Definition 3.1.** The bounded real generalised singular perturbation of stable, min-
 11 imal quadruple (A, B, C, D) , for $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$, is given by (2.5) when
 12 (A, B, C, D) is bounded real balanced, provided that it is well-defined.

13 Our two main results of this section are stated and proven next. They parallel the
 14 results in Section 2: the first contains state-space properties of the bounded real
 15 GSPA and the second contains a frequency domain error bound.

16 **Theorem 3.2.** *Given $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ and stable, minimal, and bounded real*
 17 *balanced quadruple (A, B, C, D) , assume that the bounded real singular values are*
 18 *simple. Then $(A_\xi, B_\xi, C_\xi, D_\xi)$, the bounded real generalised singular perturbation*
 19 *approximation of order $r \in \underline{n-1}$, is well-defined and the following statements hold.*

- 20 (i) $(A_\xi, B_\xi, C_\xi, D_\xi)$ is bounded real, and is bounded real balanced if $\xi \in i\mathbb{R}$.
- 21 (ii) A_ξ is Hurwitz.
- 22 (iii) If (A, B, C, D) is strictly bounded real, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and
 23 strictly bounded real.

24 Special cases of the above theorem appear in [37, Theorem 2] and [33, Theorem 2
 25 (a)], corresponding to the cases $\xi = \infty$ (the bounded real balanced truncation) and
 26 $\xi = 0$ (the bounded real singular perturbation approximation), respectively. Even
 27 in these special cases, the claim in statement (iii) above that strict bounded realness
 28 is preserved in the respective truncations does not appear in [37] or [33].

29 **Theorem 3.3.** *Let $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times m})$ be rational and bounded real with simple*
 30 *bounded real singular vales $(\sigma_j)_{j=1}^n$, ordered as in (2.3), let $r \in \underline{n-1}$ and $\xi \in \mathbb{C}$*
 31 *with $\operatorname{Re}(\xi) \geq 0$. Then there exists a rational, bounded real $\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times m})$*
 32 *which has a state-space realisation of dimension r , such that (2.7) holds and*

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^n \sigma_j. \quad (3.3)$$

33 If $\|\mathbf{G}\|_{H^\infty} < 1$, then \mathbf{G}_r^ξ may be chosen with the above properties and, additionally,
 34 to have McMillan degree r and $\|\mathbf{G}_r^\xi\|_{H^\infty} < 1$.

1 The next result pertains to existence and approximation of so-called spectral factors,
 2 and spectral “sub”-factors, particularly of reduced order transfer functions obtained
 3 by bounded real GSPA. Here \mathbf{H}^* denotes $s \mapsto (\mathbf{H}(s))^*$ for matrix-valued rational
 4 functions \mathbf{H} of a complex variable.

5 **Proposition 3.4.** *Imposing the notation and assumptions of Theorem 3.3, the*
 6 *following statements hold.*

(i) *There exist rational $\mathbf{R} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$, $\mathbf{S} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times p})$ such that*

$$I - \mathbf{G}^* \mathbf{G} = \mathbf{R}^* \mathbf{R} \quad \text{and} \quad I - \mathbf{G} \mathbf{G}^* = \mathbf{S} \mathbf{S}^* \quad \text{on } i\mathbb{R}.$$

7 (ii) *If $\xi \in i\mathbb{R}$, then there exist rational $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$, $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times p})$*
 8 *such that*

$$I - (\mathbf{G}_r^\xi)^* \mathbf{G}_r^\xi = (\mathbf{R}_r^\xi)^* \mathbf{R}_r^\xi \quad \text{and} \quad I - \mathbf{G}_r^\xi (\mathbf{G}_r^\xi)^* = \mathbf{S}_r^\xi (\mathbf{S}_r^\xi)^* \quad \text{on } i\mathbb{R}, \quad (3.4)$$

9 *and*

$$\max \left\{ \left\| \begin{pmatrix} \mathbf{G} - \mathbf{G}_r^\xi \\ \mathbf{R} - \mathbf{R}_r^\xi \end{pmatrix} \right\|_{H^\infty}, \left\| \begin{pmatrix} \mathbf{G} - \mathbf{G}_r^\xi & \mathbf{S} - \mathbf{S}_r^\xi \end{pmatrix} \right\|_{H^\infty} \right\} \leq 2 \sum_{j=r+1}^n \sigma_j, \quad (3.5)$$

10 *so that in particular*

$$\|\mathbf{R} - \mathbf{R}_r^\xi\|_{H^\infty}, \|\mathbf{S} - \mathbf{S}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^n \sigma_j. \quad (3.6)$$

11 *The spectral factors \mathbf{R}_r^ξ and \mathbf{S}_r^ξ have state-space realisations with the same*
 12 *dimension as those for \mathbf{G}_r^ξ and may be chosen with the interpolation property*

$$\mathbf{R}(\xi) = \mathbf{R}_r^\xi(\xi) \quad \text{and} \quad \mathbf{S}(\xi) = \mathbf{S}_r^\xi(\xi). \quad (3.7)$$

13 (iii) *If $\xi \in \mathbb{C}_0$, then there exist rational $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$, $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times p})$,*
 14 *such that properties (3.5)–(3.7) from statement (ii) hold, and (3.4) is replaced*
 15 *by*

$$I - (\mathbf{G}_r^\xi)^* \mathbf{G}_r^\xi \geq (\mathbf{R}_r^\xi)^* \mathbf{R}_r^\xi \quad \text{and} \quad I - \mathbf{G}_r^\xi (\mathbf{G}_r^\xi)^* \geq \mathbf{S}_r^\xi (\mathbf{S}_r^\xi)^* \quad \text{on } i\mathbb{R}. \quad (3.8)$$

16 **4. Positive real generalised SPA.** In this section we define the positive real
 17 GSPA of a quadruple with positive real transfer function, and show that it gives
 18 rise to a positive real reduced order system, with properties including the point
 19 interpolation (2.7) and error bounds. Recall that positive realness is a property of
 20 “square” systems, meaning the input and output spaces have the same dimension,
 21 $m = p$, and that a rational, $\mathbb{C}^{m \times m}$ -valued function \mathbf{G} is said to be positive real if

$$\operatorname{Re} \mathbf{G}(s) = \mathbf{G}(s) + [\mathbf{G}(s)]^* \geq 0, \quad \forall s \in \mathbb{C}_0 \setminus \Delta, \quad (4.1)$$

22 where Δ is the set of poles of \mathbf{G} . The assumption that \mathbf{G} is rational implies that \mathbf{G}
 23 is analytic on $\mathbb{C}_0 \setminus \Delta$, and it is well-known (see [20, Proposition 3.3]) that analyticity
 24 and the positive realness condition (4.1) together imply that \mathbf{G} in fact has no poles
 25 in \mathbb{C}_0 , and hence $\mathbf{G} \in H(\mathbb{C}_0, \mathbb{C}^{m \times m})$. Rational positive real functions may have
 26 simple imaginary axis poles, such as $s \mapsto 1/s$, and need not be proper, such as
 27 $s \mapsto s$.

28 Positive realness is the frequency domain term for systems which are called impedance
 29 passive, or sometimes just passive, in the time domain. For scalar, rational func-
 30 tions, the terms positive and positive real were introduced in [4], with the former
 31 used for functions which satisfy (4.1), and the latter for functions which satisfy (4.1)

1 and are also real on the real axis. As with bounded realness, although many phys-
 2 ically motivated transfer functions are real on the real axis, we do not impose this
 3 assumption simply because it is not required. However, we adopt the convention
 4 of calling such functions positive real, which agrees with much existing literature
 5 and as it captures that the real part of the function under consideration is positive
 6 (non-negative, to be precise). Positive realness and bounded realness are related
 7 via the mapping which goes by the name of the diagonal transformation, (external)
 8 Cayley transform or Möbius transform, see [19, Ch. 7], [34, Ch. 5] or [46], which we
 9 exploit in the present section to make use of the material established previously.

10 Positive real balanced truncation, proposed in [9] and further developed in [26], and
 11 positive real singular perturbation approximation, proposed in [33], are defined in
 12 the same spirit as their bounded real counterparts, where now extremal solutions
 13 of the primal and dual Positive Real Lur'e equations (or Riccati equations) are bal-
 14 anced. The theoretical result underpinning the process is the Positive Real Lemma.
 15 We note the potential confusion between the original nomenclature 'balanced sto-
 16 chastic truncation' and the more recent 'positive real balanced truncation', see [21,
 17 Remark 1]. As with the bounded real case, there are a myriad of references to these
 18 model reduction approaches for positive real systems, including those cited above
 19 and [3, 18, 19, 23, 22]. For brevity, here we describe only the key aspects which we
 20 shall require to define the positive real GSPA and establish its properties.

21 To that end, recall that if the stable, minimal quadruple (A, B, C, D) is positive
 22 real, then there exist P_m and P_M , positive definite solutions of the Positive Real
 23 Lur'e equations

$$\left. \begin{aligned} A^*Z + ZA &= -K^*K, \\ ZB - C^* &= -K^*W, \\ D + D^* &= W^*W, \end{aligned} \right\} \quad (4.2)$$

24 (with variable Z), for some $K \in \mathbb{C}^{m \times n}$ and $W \in \mathbb{C}^{m \times m}$, which are extremal in the
 25 sense that any other positive semi-definite solution P of (4.2) satisfies $P_m \leq P \leq$
 26 P_M . It is straightforward to show that P_M^{-1} is also equal to the minimal solution
 27 (in the previous sense) of the dual Positive Real Lur'e equations

$$\left. \begin{aligned} AZ + ZA^* &= -LL^*, \\ ZC^* - B^* &= -LX^*, \\ D^* + D &= XX^*, \end{aligned} \right\} \quad (4.3)$$

28 (also with variable Z) for some $L \in \mathbb{C}^{n \times m}$ and $X \in \mathbb{C}^{m \times m}$. We say that (A, B, C, D)
 29 is positive real balanced if

$$P_m = P_M^{-1} = \Sigma.$$

30 In particular, when (A, B, C, D) is positive real balanced, then Σ is a solution of
 31 both (4.2) and (4.3). The positive real singular values, denoted $(\sigma_k)_{k=1}^n$, are the
 32 nonnegative square roots of the eigenvalues of $P_m P_M^{-1}$, although like bounded real
 33 singular values, they are called characteristic values by some authors, see [42].

34 **Definition 4.1.** The positive real generalised singular perturbation of a stable,
 35 minimal quadruple (A, B, C, D) , for $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$, is given by (2.5) when
 36 (A, B, C, D) is positive real balanced, provided that it is well-defined.

37 Our two main results of this section are stated and proven next. They parallel
 38 the results in Section 3: the first contains state-space properties of the positive

1 real GSPA and the second contains frequency domain properties and error bounds.
 2 Adopting the nomenclature convention used in [20], we say that the rational, $\mathbb{C}^{m \times m}$ -
 3 valued function \mathbf{G} is strongly positive real if

$$\operatorname{Re} \mathbf{G}(s) = \mathbf{G}(s) + [\mathbf{G}(s)]^* \geq \delta I, \quad \forall s \in \mathbb{C}_0 \setminus \Delta,$$

4 for some $\delta > 0$, and where Δ denotes the set of poles of \mathbf{G} . Strongly positive real
 5 functions are clearly positive real.

6 **Theorem 4.2.** *Given $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ and stable, minimal, and positive real
 7 balanced quadruple (A, B, C, D) , assume that the positive real singular values are
 8 simple. Then $(A_\xi, B_\xi, C_\xi, D_\xi)$, the positive real generalised singular perturbation
 9 approximation of order $r \in \underline{n-1}$, is well-defined and the following statements hold.*

- 10 (i) $(A_\xi, B_\xi, C_\xi, D_\xi)$ is positive real, and is positive real balanced if $\xi \in i\mathbb{R}$.
- 11 (ii) A_ξ is Hurwitz.
- 12 (iii) If (A, B, C, D) is strongly positive real, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and
 13 strongly positive real.

14 **Theorem 4.3.** *Let $\mathbf{G} \in H(\mathbb{C}_0, \mathbb{C}^{m \times m})$ be proper, rational, and positive real with
 15 simple positive real singular vales $(\sigma_j)_{j=1}^n$, ordered as in (2.3), let $r \in \underline{n-1}$ and $\xi \in$
 16 \mathbb{C} with $\operatorname{Re}(\xi) \geq 0$ which is not a pole of \mathbf{G} . Then there exists proper, rational, and
 17 positive real $\mathbf{G}_r^\xi \in H(\mathbb{C}_0, \mathbb{C}^{m \times m})$ which has a state-space realisation of dimension
 18 r , such that (2.7) holds and*

$$\hat{\delta}(\mathbf{G}, \mathbf{G}_r^\xi) \leq 2 \sum_{j=r+1}^n \sigma_j, \quad (4.4)$$

where $\hat{\delta}$ denotes the gap metric [28, p.197, p.201]. If $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$, then \mathbf{G}_r^ξ
 with the previous properties may be chosen to be in $H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$ as well, and

$$\begin{aligned} \|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} &\leq 2 \min \left\{ (1 + \|\mathbf{G}\|_{H^\infty}^2)(1 + \|\mathbf{G}_r^\xi\|_{H^\infty}), \right. \\ &\quad \left. (1 + \|\mathbf{G}\|_{H^\infty})(1 + \|\mathbf{G}_r^\xi\|_{H^\infty}^2) \right\} \sum_{j=r+1}^n \sigma_j, \end{aligned} \quad (4.5)$$

19 holds. Finally, if \mathbf{G} is strongly positive real, then \mathbf{G}_r^ξ as above may be chosen to
 20 have McMillan degree r and be strongly positive real as well.

21 In certain cases, the error bound (4.5) may be used to derive a more conservative
 22 (that is, worse), but *a priori*, bound. The reader is referred to [19, Remark 3.6.11]
 23 for more details.

24 Our final result pertains to existence of so-called spectral factors, now in the pos-
 25 itive real case, and is the positive real analogue of Proposition 3.4. Although our
 26 approach is to use the Cayley transform and Proposition 3.4, ‘natural’ error bounds
 27 in the gap metric for the distance between spectral factors and their approximations
 28 in the positive real case sadly do not seemingly follow from those in the bounded
 29 real case. For completeness, we do provide an H^∞ error bound in the special
 30 case that $\mathbf{G} \in H^\infty$ which, in keeping with the GSPA, does depend linearly on the
 31 sum of omitted singular values. The constant which appears in the bound may be
 32 somewhat conservative, however.

33 **Proposition 4.4.** *Imposing the notation and assumptions of Theorem 4.3, let Δ
 34 denote the set of poles of \mathbf{G} on $i\mathbb{R}$. The following statements hold.*

(i) There exists a proper, rational, $\mathbb{C}^{m \times m}$ -valued function \mathbf{R} such that

$$\mathbf{G} + \mathbf{G}^* = \mathbf{R}^* \mathbf{R} \quad \text{on } i\mathbb{R} \setminus \Delta.$$

(ii) If $\xi \in i\mathbb{R}$, then there exists a proper, rational $\mathbb{C}^{m \times m}$ -valued function \mathbf{R}_r^ξ such that

$$\mathbf{G}_r^\xi + (\mathbf{G}_r^\xi)^* = (\mathbf{R}_r^\xi)^* \mathbf{R}_r^\xi \quad \text{on } i\mathbb{R} \setminus \Delta.$$

The functions \mathbf{R} and \mathbf{R}_r^ξ may be chosen with the property that $\mathbf{R}(\xi) = \mathbf{R}_r^\xi(\xi)$ and, further, \mathbf{R}_r^ξ and \mathbf{G}_r^ξ have state-space realisations with the same dimension.

If $\mathbf{G} \in H^\infty$, then \mathbf{R} and \mathbf{R}_r^ξ may be chosen to belong to H^∞ as well. In this case it follows that

$$\begin{aligned} \|\mathbf{R} - \mathbf{R}_r^\xi\|_{H^\infty} &\leq \min \left\{ 2a \|\mathbf{R}(I + \mathbf{G})^{-1}\|_{H^\infty} + \sqrt{2} \|I + \mathbf{G}_r^\xi\|_{H^\infty}, \right. \\ &\quad \left. 2a \|\mathbf{R}_r^\xi(I + \mathbf{G}_r^\xi)^{-1}\|_{H^\infty} + \sqrt{2} \|I + \mathbf{G}\|_{H^\infty} \right\} \sum_{j=r+1}^n \sigma_j, \end{aligned}$$

where

$$a := \min \left\{ (1 + \|\mathbf{G}\|_{H^\infty}^2)(1 + \|\mathbf{G}_r^\xi\|_{H^\infty}), (1 + \|\mathbf{G}\|_{H^\infty})(1 + \|\mathbf{G}_r^\xi\|_{H^\infty}^2) \right\}.$$

5. Examples.

Example 5.1. Let \mathbf{G} denote the strictly bounded real transfer function

$$s \mapsto \mathbf{G}(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)(s+5)},$$

considered in [37, Section V] and then [33, Example 1]. A minimal realisation of \mathbf{G} is

$$A = \begin{pmatrix} -12 & -5.875 & -3.75 \\ 8 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = (1 \quad 0.375 \quad 0.125), \quad D = 0,$$

and the bounded real singular values are

$$\sigma_1 = 5.21 \times 10^{-2}, \quad \sigma_2 = 3.61 \times 10^{-2}, \quad \sigma_3 = 6.35 \times 10^{-4}.$$

Figures 5.1 and 5.2 plot the combined error

$$\left\| \begin{pmatrix} \mathbf{G}(s) - \mathbf{G}_r^{\xi_j}(s) \\ \mathbf{R}(s) - \mathbf{R}_r^{\xi_j}(s) \end{pmatrix} \right\|_2,$$

against real $s > 0$ for several $\xi_j > 0$, for the cases $r = 1$ and $r = 2$, respectively. Here \mathbf{R} is a spectral factor for $I - \mathbf{G}^* \mathbf{G}$ and \mathbf{R}_r^ξ is a sub-spectral factor for $I - (\mathbf{G}_r^\xi)^* \mathbf{G}_r^\xi$, in the sense of statement (iii) of Proposition 3.4. We see in the plots the interpolation properties (2.7) and (3.7) holding. As expected from inspection of the bounded real singular values — the first two are of the same order — the errors are much smaller when $r = 2$, compare the y -axes of Figures 5.1 and 5.2. Figure 5.3 plots the error $|\mathbf{G}(\omega i) - \mathbf{G}_r^{\xi_j}(\omega i)|$ on an interval of the imaginary axis. Recall that the infinity norm error $\|\mathbf{G} - \mathbf{G}_r^{\xi_j}\|_{H^\infty}$ will be achieved at some such ω . Observe that the choice of point of interpolation ξ_j seemingly leads to a trade-off between the error of the approximations at $\omega = 0$ (the steady state gain) and $\omega = \infty$ (the feedthrough). \square

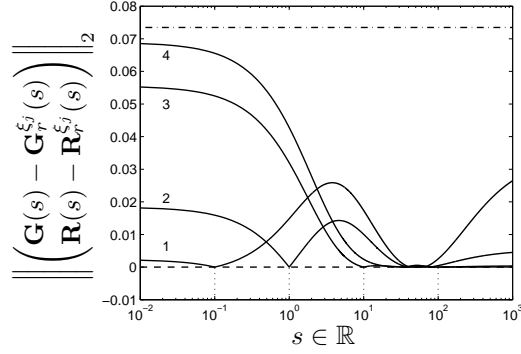


FIGURE 5.1. Semi-log plot of combined errors on the real axis for the bounded real GSPA from Example 5.1, with $r = 1$. The lines numbered 1–4 correspond to $\xi_1 = 0.1$, $\xi_2 = 1$, $\xi_3 = 10$ and $\xi_4 = 100$, respectively. Note the interpolation properties (2.7) and (3.7) hold and are highlighted with vertical dotted lines. The dashed dotted line is the bound (3.3).

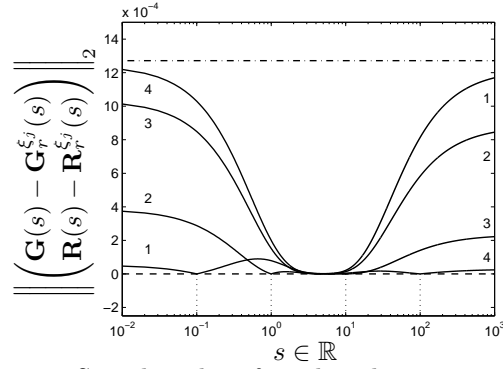


FIGURE 5.2. Semi-log plot of combined errors on the real axis for the bounded real GSPA from Example 5.1, with $r = 2$. The lines numbered 1–4 correspond to $\xi_1 = 0.1$, $\xi_2 = 1$, $\xi_3 = 10$ and $\xi_4 = 100$, respectively. Note the interpolation properties (2.7) and (3.7) hold and are highlighted with vertical dotted lines. The dashed dotted line is the error bound (3.3).

- 1 *Example 5.2.* The paper [38, Section V] considers model reduction of RC ladder
- 2 circuit arrangements. The first circuit in that paper, which we consider here, has
- 3 two current sources which gives rise to MIMO control system with the state-space
- 4 realisation

$$\left. \begin{aligned} A &= \begin{pmatrix} -\frac{3}{2\mathcal{R}\mathcal{C}} & \frac{1}{2\mathcal{R}\mathcal{C}} & 0 & 0 \\ \frac{1}{\mathcal{R}\mathcal{C}} & -\frac{1}{\mathcal{R}\mathcal{C}} & \frac{1}{\mathcal{R}\mathcal{C}} & 0 \\ 0 & \frac{1}{\mathcal{R}\mathcal{C}} & -\frac{1}{\mathcal{R}\mathcal{C}} & \frac{1}{\mathcal{R}\mathcal{C}} \\ 0 & 0 & \frac{1}{\mathcal{R}\mathcal{C}} & -\frac{1}{2\mathcal{R}\mathcal{C}} \end{pmatrix}, & B &= \begin{pmatrix} -\frac{1}{\mathcal{C}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{\mathcal{C}} \end{pmatrix}, \\ C &= B^T, & D &= \begin{pmatrix} \frac{\mathcal{R}}{2} & 0 \\ 0 & \frac{\mathcal{R}}{2} \end{pmatrix}. \end{aligned} \right\} \quad (5.1)$$

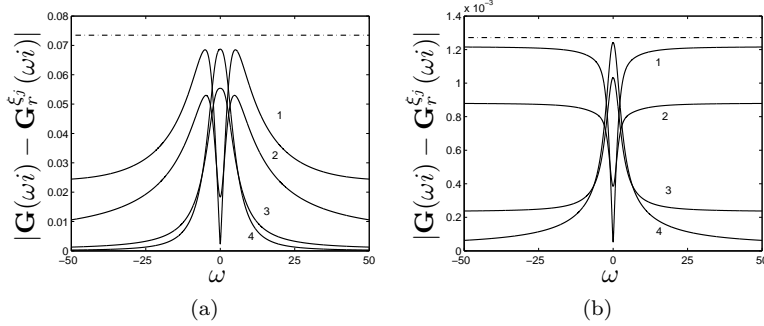


FIGURE 5.3. Plots of errors on the imaginary axis for the bounded real GSPA from Example 5.1, with $r = 1$ and $r = 2$ in panels (a) and (b), respectively. The lines numbered 1–4 correspond to $\xi_1 = 0.1$, $\xi_2 = 1$, $\xi_3 = 10$ and $\xi_4 = 100$, respectively, and are symmetric around $\omega = 0$. The dashed dotted lines are the bounds (3.3).

- 1 Here the terms \mathcal{R} and \mathcal{C} are positive parameters (resistances and capacitances,
2 respectively). The inputs are currents at the sources, the outputs are voltages
3 at the sources, and the state variables are voltages at the capacitors. We refer the
4 reader to [38, Section V] for more details. The quadruple in (5.1) is strongly positive
5 real, as $A + A^* \leq 0$, $B = C^*$ and $D + D^* > 0$. With $\mathcal{R} = \mathcal{C} = 1$, the positive real
6 singular values are (to three significant figures)

$$\sigma_1 = 0.153, \quad \sigma_2 = 0.0870 \quad \sigma_3 = 0.0190 \quad \sigma_4 = 0.00190,$$

- 7 which, note, are different to the *Hankel* singular values of (5.1) computed in [38].
8 Figure 5.4 plots the error $\|\mathbf{G}(s) - \mathbf{G}_r^\xi(s)\|_2$, where \mathbf{G}_r^ξ now denotes the positive real
GSPA, against real $s > 0$ for fixed $\xi = 10$, for $r \in \{1, 2, 3\}$.

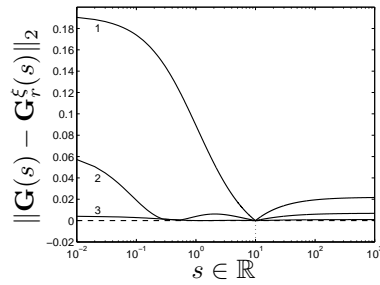


FIGURE 5.4. Semi-log plot of combined errors on the real axis for the positive real GSPA from Example 5.2, with $\xi = 10$. The lines numbered 1–3 correspond to $r \in \{1, 2, 3\}$ respectively. Note the interpolation property (2.7) holds.

- 10 The circuit in [38, Section V] may be easily be extended by adding identical “rungs”
11 of the ladder, with each capacitor adding another state variable. As an illustrative
12 example, we chose $N = 15$ capacitors, giving 15 states, with the same inputs and
13 outputs as before. It is readily established from Kirchoff’s laws and elementary

1 circuit theory that the resulting matrix A has the same tri-banded structure as that
 2 in (5.1). The new B matrix has the same first and last row as that in (5.1), but with
 3 more rows of zeros in the middle. Further, $C = B^T$ still holds and D is unchanged.
 4 Fixing $\xi = 10$, we computed the error in the gap metric between \mathbf{G} and \mathbf{G}_r^ξ for
 5 $r \in \{1, 2, \dots, 13\}$, as well as the error bounds from (4.4). The results are plotted
 6 on a semi-log axis in Figure 5.5. Although the errors are larger than the bound for
 7 $r \geq 10$, we expect that this is a consequence of the Matlab's function `gapmetric`
 8 maximal error tolerance of 1×10^{-5} . \square

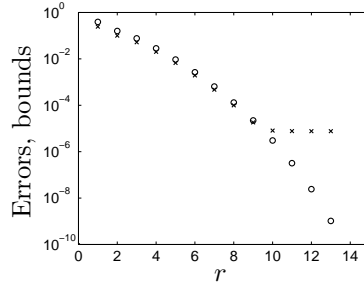


FIGURE 5.5. Semi-log plot of gap metric error $\hat{\delta}(\mathbf{G}, \mathbf{G}_r^\xi)$ (crosses) and error bounds (4.4) (circles) for extended circuit model from Example 5.2. Here $\xi = 10$.

9 **6. Proofs of results in Sections 3 and 4.** We divide the section into two sub-
 10 sections, considering the bounded real and positive real cases separately.

11 **6.1. The bounded real generalised singular perturbation approximation.**

12 In order to prove Theorems 3.2 and 3.3, we draw on the material presented in
 13 Section 2, and also require three technical lemmas, stated and proven first.

14 **Lemma 6.1.** *If stable (A, B, C, D) with transfer function \mathbf{G} and $\Sigma = \Sigma^* \geq 0$ are*
 15 *such that*

$$\left. \begin{aligned} A^*\Sigma + \Sigma A + C^*C &= -K^*K - P^*P \\ \Sigma B + C^*D &= -K^*W - P^*Q \\ I - D^*D &= W^*W + Q^*Q \end{aligned} \right\}, \quad (6.1)$$

16 *hold for some $K, P \in \mathbb{C}^{m \times n}$ and $Q, W \in \mathbb{C}^{m \times m}$, then*

- 17 (i) (A, B, C, D) is bounded real.
 18 (ii) $\mathbf{R} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{2m \times m})$ with realisation $(A, B, \begin{bmatrix} K \\ P \end{bmatrix}, \begin{bmatrix} W \\ Q \end{bmatrix})$ is a spectral factor
 19 for $I - \mathbf{G}^*\mathbf{G}$ in the sense that

$$I - (\mathbf{G}(s))^*\mathbf{G}(s) = (\mathbf{R}(s))^*\mathbf{R}(s) \quad \forall s \in i\mathbb{R}.$$

20 *Further, if the dual equations*

$$\left. \begin{aligned} A\Sigma + \Sigma A^* + BB^* &= -LL^* - RR^* \\ \Sigma C^* + BD^* &= -LX^* - RS^* \\ I - DD^* &= XX^* + SS^* \end{aligned} \right\}, \quad (6.2)$$

21 *hold for some $L, R \in \mathbb{C}^{n \times p}$ and $X, S \in \mathbb{C}^{p \times p}$, then*

- 1 (iii) $\mathbf{S} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times 2p})$ with realisation $(A, [B \ R], C, [X \ S])$ is a spectral factor
 2 for $I - \mathbf{G}\mathbf{G}^*$ in the sense that

$$I - \mathbf{G}(s)(\mathbf{G}(s))^* = \mathbf{S}(s)(\mathbf{S}(s))^* \quad \forall s \in i\mathbb{R}.$$

- 3 Observe that in the above lemma, if $P = 0$ and $Q = 0$, then (A, B, K, W) is a
 4 realisation of a spectral factor \mathbf{R} . Similarly, if $R = 0$ and $S = 0$, then (A, L, C, X)
 5 is a realisation of a spectral factor \mathbf{S} .

Proof of Lemma 6.1: To prove statement (i), let $x^0 \in \mathbb{C}^n$, u be a continuous control and $x = x(\cdot; u, x^0)$ the corresponding differentiable state. From (6.1) we have that for all $\tau \geq 0$

$$\begin{aligned} \frac{d}{d\tau} \langle x(\tau), \Sigma x(\tau) \rangle + \|y(\tau)\|^2 - \|u(\tau)\|^2 \\ = \left\langle \begin{pmatrix} A^* \Sigma + \Sigma A + C^* C & \Sigma B + C^* D \\ B^* \Sigma + D^* C & D^* D - I \end{pmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}, \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \right\rangle \\ \leq - \left\| \begin{pmatrix} K & W \end{pmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \right\|^2 \\ \leq 0. \end{aligned} \quad (6.3)$$

- 6 Integrating both sides of (6.3) between 0 and $t \geq 0$ gives

$$\int_0^t \frac{d}{d\tau} \langle x(\tau), \Sigma x(\tau) \rangle + \|y(\tau)\|^2 - \|u(\tau)\|^2 d\tau \leq 0 \quad \forall t \geq 0,$$

- 7 whence

$$\int_0^t \|y(\tau)\|^2 - \|u(\tau)\|^2 d\tau \leq \langle x^0, \Sigma x^0 \rangle - \langle x(t), \Sigma x(t) \rangle \quad \forall t \geq 0. \quad (6.4)$$

- 8 By a continuity and density argument, the inequality (6.4) holds for all $u \in L^2$ with
 9 corresponding continuous state x . With zero initial state $x^0 = 0$, it follows that
 10 the input u and output y satisfy $\|y\|_{L^2} \leq \|u\|_{L^2}$, and hence (A, B, C, D) is bounded
 11 real.

Statement (ii) follows from an elementary calculation using the equalities in (6.1). Indeed, let $s \in i\mathbb{R}$ and consider

$$\begin{aligned} I - (\mathbf{G}(s))^* \mathbf{G}(s) &= I - (D + C(sI - A)^{-1}B)^*(D + C(sI - A)^{-1}B) \\ &= I - D^*D - D^*C(sI - A)^{-1}B - B^*(sI - A)^{-*}C^*D \\ &\quad - B^*(sI - A)^{-*}C^*C(sI - A)^{-1}B \\ &= W^*W^* + Q^*Q + (B^*\Sigma + W^*K + Q^*P)(sI - A)^{-1}B \\ &\quad + B^*(sI - A)^{-*}(\Sigma B + K^*W + P^*Q) \\ &\quad + B^*(sI - A)^{-*}(A^*\Sigma + \Sigma A + K^*K + P^*P)(sI - A)^{-1}B \\ &= W^*W^* + Q^*Q + (W^*K + Q^*P)(sI - A)^{-1}B \\ &\quad + B^*(sI - A)^{-*}(K^*W + P^*Q) \\ &\quad + B^*(sI - A)^{-*}(K^*K + P^*P)(sI - A)^{-1}B \\ &= \left(\begin{pmatrix} W \\ Q \end{pmatrix} + \begin{pmatrix} K \\ P \end{pmatrix} (sI - A)^{-1}B \right)^* \left(\begin{pmatrix} W \\ Q \end{pmatrix} + \begin{pmatrix} K \\ P \end{pmatrix} (sI - A)^{-1}B \right) \\ &= (\mathbf{R}(s))^* \mathbf{R}(s). \end{aligned}$$

Statement (iii) is proven similarly, only instead using the equalities in (6.2). The details are omitted. \square

For $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ and stable (A, B, C, D) , set

$$\left. \begin{aligned} \mathcal{A} &:= (A - \xi I)^{-1}, & \mathcal{B} &:= (A - \xi I)^{-1}B, \\ \mathcal{C} &:= C(A - \xi I)^{-1}, & \mathcal{D} &:= D - C(A - \xi I)^{-1}B, \end{aligned} \right\} \quad (6.5)$$

which are well-defined and based on the reciprocal transformation. For given $r \in \underline{n-1}$, let the decomposition $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1)$ be analogous to those in (2.4). The next lemma describes properties of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and relationships with (A_ξ, B_ξ, C_ξ) .

Lemma 6.2. *For $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ and stable (A, B, C) , assume that (A_ξ, B_ξ, C_ξ) given by (2.5) is well-defined and let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be given by (6.5). The following statements hold.*

- (1) *If (A, B, C) is controllable or observable, then $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ has the same respective property.*
- (2) *$\xi \notin \sigma(A_\xi)$ so that $A_\xi - \xi I$ is invertible and*

$$\mathcal{A}_{11} = (A_\xi - \xi I)^{-1}, \quad \mathcal{B}_1 = (A_\xi - \xi I)^{-1}B_\xi, \quad \mathcal{C}_1 = C_\xi(A_\xi - \xi I)^{-1}. \quad (6.6)$$

- (3) *If $M \in \mathbb{C}^{n \times n}$ is Hurwitz and $\xi \in \mathbb{C}_0$, then $\sigma((M - \xi I)^{-1}) \subseteq \mathbb{E}_\xi$, where*

$$\mathbb{E}_\xi := \{s \in \mathbb{C} : |s + 1/(2\operatorname{Re}(\xi))| < 1/(2\operatorname{Re}(\xi))\}. \quad (6.7)$$

If $\xi \in i\mathbb{R}$, then $(M - \xi I)^{-1}$ is Hurwitz.

- (4) *\mathcal{A} in (6.5) is Hurwitz.*

Proof. (1): We use the Hautus criterion for observability. Assume that $v \in \mathbb{C}^n$ is such that $\mathcal{A}v = \lambda v$ and $\mathcal{C}v = 0$. Since \mathcal{A} is invertible, if $\lambda = 0$, then $v = 0$ and there is nothing to prove. If $\lambda \neq 0$, then rearranging gives $Av = (\xi + 1/\lambda)v$ and

$$0 = \mathcal{C}v = C(A - \xi I)^{-1}v = \lambda \mathcal{C}v,$$

so that $\mathcal{C}v = 0$. As the pair (C, A) is observable, it follows that $v = 0$, and thus the pair $(\mathcal{C}, \mathcal{A})$ is also observable. The proof of the controllability claim is similar, and so the details are omitted.

(2): We prove that $\xi \notin \sigma(A_\xi)$ by contraposition. If $v \neq 0$ and $\xi \in \mathbb{C}$ are such that $A_\xi v = \xi v$, then

$$\begin{aligned} A \begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix} \\ &= \begin{pmatrix} A_\xi v \\ \xi v \end{pmatrix} = \xi \begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix}, \end{aligned}$$

and we conclude that $\xi \in \sigma(A)$. The claim now follows as A is assumed Hurwitz, but $\operatorname{Re}(\xi) \geq 0$.

The equalities in (6.6) follow from block-wise matrix inversion and the definitions in (2.5) and (6.5).

(3): Let $\lambda \in \sigma((M - \xi I)^{-1})$ (so that necessarily $\lambda \neq 0$). Then $\xi + 1/\lambda \in \sigma(M)$, so that

$$\begin{aligned} \operatorname{Re}(\xi + 1/\lambda) < 0 &\Rightarrow \operatorname{Re}(\xi) < -\operatorname{Re}(1/\lambda) = \frac{-\operatorname{Re}(\lambda)}{|\lambda|^2} \\ &\Rightarrow \operatorname{Re}(\lambda) < -\operatorname{Re}(\xi)|\lambda|^2. \end{aligned} \quad (6.8)$$

1 If $\xi \in \mathbb{C}_0$, then (6.8) gives that $\lambda \in \mathbb{E}_\xi$, as required. If $\operatorname{Re}(\xi) = 0$, then (6.8) now
2 yields that $\operatorname{Re}(\lambda) < 0$.

3 (4): Follows from (3), upon noticing that $\mathbb{E}_\xi \subset \mathbb{C}_0$. □

4 In the sequel we shall require the simple observation that for $\xi \in \mathbb{C}_0$

$$\lambda \in \partial\mathbb{E}_\xi \iff \operatorname{Re}(\lambda) = -\operatorname{Re}(\xi)|\lambda|^2, \quad (6.9)$$

5 where $\partial\mathbb{E}_\xi$ denotes the boundary of \mathbb{E}_ξ — the circle in the complex plane with
6 radius $1/(2\operatorname{Re}(\xi))$ and centre $-1/(2\operatorname{Re}(\xi))$.

7 **Lemma 6.3.** *Given $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$, suppose that stable (A, B, C, D) has*
8 *transfer function \mathbf{G} . Define \mathbf{G}_r^ξ , \mathbf{H} and \mathbf{H}_r as the transfer functions with reali-*
9 *sations $(A_\xi, B_\xi, C_\xi, D_\xi)$, $(\mathcal{A}, \mathcal{B}, -\mathcal{C}, \mathcal{D})$ and $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$, respectively. Assume*
10 *that $\sigma(\mathcal{A}_{11}) \subseteq \mathbb{E}_\xi$ if $\xi \in \mathbb{C}_0$, or \mathcal{A}_{11} is Hurwitz if $\xi \in i\mathbb{R}$. Then*

$$\mathbf{G}(z) = \mathbf{H}\left(\frac{1}{z - \xi}\right) \quad \forall z \in \mathbb{C}_0, \operatorname{Re}(z) \geq 0, \quad z \neq \xi, \quad (6.10)$$

11 and

$$\mathbf{G}_r^\xi(z) = \mathbf{H}_r\left(\frac{1}{z - \xi}\right) \quad \forall z \in \mathbb{C}_0, \operatorname{Re}(z) \geq 0, \quad z \neq \xi. \quad (6.11)$$

Proof. Invoking Lemma 6.2, as A is Hurwitz, either $\sigma(\mathcal{A}) \subseteq \mathbb{E}_\xi$ or \mathcal{A} is Hurwitz, depending on whether $\xi \in \mathbb{C}_0$ or $\xi \in i\mathbb{R}$, respectively. For $z \in \mathbb{C}$, $\operatorname{Re}(z) \geq 0$ and $z \neq 0$, we compute that

$$\begin{aligned} \mathbf{G}(\xi + 1/z) &= D + C((\xi + 1/z)I - A)^{-1}B = D + C(1/zI - (A - \xi I))^{-1}B \\ &= D - Cz(zI - (A - \xi I)^{-1})^{-1}(A - \xi I)^{-1}B \\ &= D - C(A - \xi I)^{-1}B - C(A - \xi I)^{-1}(zI - (A - \xi I)^{-1})^{-1}(A - \xi I)^{-1}B \\ &= \mathcal{D} - \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} \\ &= \mathbf{H}(z). \end{aligned} \quad (6.12)$$

Similarly, using the relationships (6.6), we have that

$$\begin{aligned} \mathbf{H}_r(z) &= \mathcal{D} - \mathcal{C}_1(zI - \mathcal{A}_{11})^{-1}\mathcal{B}_1 \\ &= D - C_\xi(A_\xi - \xi I)^{-1}(zI - (A_\xi - \xi I)^{-1})^{-1}(A_\xi - \xi I)^{-1}B_\xi \\ &= D + C_\xi(A_\xi - \xi I)^{-1}B_\xi + C_\xi((\xi + 1/z)I - A_\xi)^{-1}B_\xi \\ &= \mathbf{G}_r^\xi(\xi + 1/z), \end{aligned} \quad (6.13)$$

where we have used (2.7) to infer that

$$\begin{aligned} D + C_\xi(A_\xi - \xi I)^{-1}B_\xi &= D - C(A - \xi I)^{-1}B + C_\xi(A_\xi - \xi I)^{-1}B_\xi \\ &= \mathbf{G}(\xi) - (\mathbf{G}_r^\xi(\xi) - D_\xi) \\ &= D_\xi. \end{aligned}$$

Therefore, combining (6.12) and (6.13) with a change of variables yields (6.10) and (6.11), respectively. \square

Proof of Theorem 3.2. Let $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$. An application of [40, Theorem 3.2] to the first equations in (3.1) and (3.2), both with $Z = \Sigma$, shows that A_{22} is Hurwitz, so that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well-defined. Elementary calculations using the definitions of $(A_\xi, B_\xi, C_\xi, D_\xi)$ in (2.5) and the equalities (3.1) and (3.2) considered block wise show that

$$\left. \begin{aligned} A_\xi^* \Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi &= -K_\xi^* K_\xi - 2\operatorname{Re}(\xi) A_{21}^* \phi^* \Sigma_2 \phi A_{21}, \\ \Sigma_1 B_\xi + C_\xi^* D_\xi &= -K_\xi^* W_\xi - 2\operatorname{Re}(\xi) A_{21}^* \phi^* \Sigma_2 \phi B_2, \\ I - D_\xi^* D_\xi &= W_\xi^* W_\xi + 2\operatorname{Re}(\xi) B_2^* \phi^* \Sigma_2 \phi B_2, \end{aligned} \right\} \quad (6.14)$$

and

$$\left. \begin{aligned} A_\xi \Sigma_1 + \Sigma_1 A_\xi^* + B_\xi B_\xi^* &= -L_\xi L_\xi^* - 2\operatorname{Re}(\xi) A_{12} \phi \Sigma_2 \phi^* A_{12}^*, \\ \Sigma_1 C_\xi^* + B_\xi D_\xi^* &= -L_\xi X_\xi^* - 2\operatorname{Re}(\xi) A_{12} \phi \Sigma_2 \phi^* C_2^*, \\ I - D_\xi D_\xi^* &= X_\xi X_\xi^* + 2\operatorname{Re}(\xi) C_2 \phi \Sigma_2 \phi^* C_2^*, \end{aligned} \right\} \quad (6.15)$$

where $\phi := (\xi I - A_{22})^{-1}$ and

$$\left. \begin{aligned} K_\xi &:= K_1 + K_2 \phi A_{21}, & W_\xi &:= W + K_2 \phi B_2, \\ L_\xi &:= L_1 + A_{12} \phi L_2, & X_\xi &:= X + C_2 \phi L_2. \end{aligned} \right\} \quad (6.16)$$

In light of Lemma 6.1 and (6.14), it follows that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is bounded real. Evidently, if $\xi \in i\mathbb{R}$, then the resulting simplification of (6.14) and (6.15) implies that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is bounded real balanced, completing the proof of statement (i).

We proceed to prove statements (ii) and (iii), treating the cases $\xi \in \mathbb{C}_0$ and $\xi \in i\mathbb{R}$ separately. Assume that $\xi \in \mathbb{C}_0$. The first equation in (6.14) implies that every eigenvalue of A_ξ has non-positive real part. Suppose that $A_\xi v = \eta i v$ for some $\eta \in \mathbb{R}$ and $v \in \mathbb{C}^r$. Forming the inner product

$$\langle (A_\xi^* \Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi) v, v \rangle,$$

and using (6.14), it follows that

$$0 \leq \|C_\xi v\|^2 = -\|K_\xi v\|^2 - 2\operatorname{Re}(\xi) \langle \Sigma_2 (\xi I - A_{22})^{-1} A_{21} v, (\xi I - A_{22})^{-1} A_{21} v \rangle \leq 0,$$

whence

$$\langle \Sigma_2 (\xi I - A_{22})^{-1} A_{21} v, (\xi I - A_{22})^{-1} A_{21} v \rangle = 0,$$

as $\operatorname{Re}(\xi) > 0$. Since $\Sigma_2 > 0$, we infer that

$$(\xi I - A_{22})^{-1} A_{21} v = 0.$$

Consequently

$$A \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \begin{pmatrix} A_\xi v \\ \xi (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \eta i \begin{pmatrix} v \\ 0 \end{pmatrix},$$

and, as A is Hurwitz, we deduce that $v = 0$. Recalling our supposition that $A_\xi v = \eta i v$, we conclude that A_ξ is Hurwitz as well.

For $\xi \in \mathbb{C}_0$, and for statement (iii), we shall require $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ defined in (6.5), which is stable by statement (3) of Lemma 6.2. Calculations starting from (3.1)

1 and (3.2) respectively show that

$$\left. \begin{aligned} \mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \mathcal{C}^*\mathcal{C} &= -\mathcal{K}^*\mathcal{K} - 2\operatorname{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{A} \\ \Sigma\mathcal{B} - \mathcal{C}^*\mathcal{D} &= \mathcal{K}^*\mathcal{W} - 2\operatorname{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{B} \\ I - \mathcal{D}^*\mathcal{D} &= \mathcal{W}^*\mathcal{W} + 2\operatorname{Re}(\xi)\mathcal{B}^*\Sigma\mathcal{B} \end{aligned} \right\}, \quad (6.17)$$

2 and

$$\left. \begin{aligned} \mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \mathcal{B}\mathcal{B}^* &= -\mathcal{L}\mathcal{L}^* - 2\operatorname{Re}(\xi)\mathcal{A}\Sigma\mathcal{A}^* \\ \Sigma(-\mathcal{C}^*) + \mathcal{B}\mathcal{D}^* &= -\mathcal{L}\mathcal{X}^* - 2\operatorname{Re}(\xi)\mathcal{A}\Sigma(-\mathcal{C})^* \\ I - \mathcal{D}\mathcal{D}^* &= \mathcal{X}\mathcal{X}^* + 2\operatorname{Re}(\xi)\mathcal{C}\Sigma\mathcal{C}^* \end{aligned} \right\}, \quad (6.18)$$

3 where

$$\mathcal{K} := K\mathcal{A}, \quad \mathcal{W} := W - K\mathcal{B}, \quad \mathcal{L} := \mathcal{A}L, \quad \text{and} \quad \mathcal{X} := X - \mathcal{C}L. \quad (6.19)$$

4 The first equations in (6.17) and (6.18) may respectively be rewritten as

$$\mathcal{A}^*\Sigma + \Sigma\mathcal{A} + (\mathcal{C}^* \quad \mathcal{K}^*) \begin{pmatrix} \mathcal{C} \\ \mathcal{K} \end{pmatrix} = -2\operatorname{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{A}, \quad (6.20)$$

5 and

$$\mathcal{A}\Sigma + \Sigma\mathcal{A}^* + (\mathcal{B} \quad \mathcal{L}) \begin{pmatrix} \mathcal{B}^* \\ \mathcal{L}^* \end{pmatrix} = -2\operatorname{Re}(\xi)\mathcal{A}\Sigma\mathcal{A}^*. \quad (6.21)$$

6 If $\xi \in i\mathbb{R}$, then a consequence of the simplification of (6.21) and (6.20) is that

$$\left(\mathcal{A}, (\mathcal{B} \quad \mathcal{L}), \begin{pmatrix} \mathcal{C} \\ \mathcal{K} \end{pmatrix} \right),$$

7 is Lyapunov balanced. An application of [40, Theorem 3.2] yields that \mathcal{A}_{11} is
 8 Hurwitz, again invoking the assumption that the singular values are simple implies
 9 that the spectra of Σ_1 and Σ_2 are disjoint. Statement (2) of Lemma 6.2 implies
 10 that $\xi \notin \sigma(A_\xi)$ and that (6.6) holds, from which it is routine to verify that A_ξ is
 11 Hurwitz, since \mathcal{A}_{11} is, and $\xi \in i\mathbb{R}$. The proof of statement (ii) is complete.

To prove statement (iii), we additionally assume that (A, B, C, D) is strictly bounded real. Suppose first that $\xi \in \mathbb{C}_0$. To establish minimality, let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be such that $A_\xi v = \lambda v$ and $C_\xi v = 0$. We compute that

$$A \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \begin{pmatrix} A_\xi v \\ \xi(\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix},$$

12 so that

$$Az = Ez,$$

13 where

$$E := \begin{pmatrix} \lambda & 0 \\ 0 & \xi \end{pmatrix} \quad \text{and} \quad z := \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix}.$$

14 An application of [40, Theorem 3.1] to the (Lyapunov) balanced realisation

$$\left(A, (B \quad L), \begin{pmatrix} C \\ K \end{pmatrix} \right),$$

15 implies that

$$\|e^{At}z\|^2 = \|e^{Et}z\|^2 = \left\| \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\xi t} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|^2 < \|z\|^2 \quad \forall t > 0,$$

16 whence

$$e^{2\operatorname{Re}(\xi)t}\|z_2\|^2 \leq e^{2\operatorname{Re}(\lambda)t}\|z_1\|^2 + e^{2\operatorname{Re}(\xi)t}\|z_2\|^2 < \|z_1\|^2 + \|z_2\|^2 \quad \forall t > 0. \quad (6.22)$$

1 Since $\xi \in \mathbb{C}_0$, the inequality (6.22) yields that

$$z_2 = (\xi I - A_{22})^{-1} A_{21} v = 0,$$

2 from which

$$\lambda v = A_\xi v = A_{11} v + A_{12} (\xi I - A_{22})^{-1} A_{21} v = A_{11} v,$$

3 and

$$0 = C_\xi v = C_1 v + C_2 (\xi I - A_{22})^{-1} A_{21} v = C_1 v.$$

4 As (A, B, C, D) is bounded real balanced and strictly bounded real, the pair (C_1, A_{11})
 5 is observable by [37, Theorem 2], and so we deduce that $v = 0$, proving that (C_ξ, A_ξ)
 6 is observable. The proof that the pair (A_ξ, B_ξ) is controllable is similar, and thus
 7 is omitted.

8 Let \mathbf{G} and \mathbf{H} be realised by (A, B, C, D) and $(\mathcal{A}, \mathcal{B}, -\mathcal{C}, \mathcal{D})$, respectively. If $\xi \in i\mathbb{R}$,
 9 then the equality (6.12) gives

$$\|\mathbf{H}\|_{H^\infty} = \sup_{z \in \mathbb{C}_0} \|\mathbf{H}(z)\|_2 = \sup_{z \in \mathbb{C}_0} \|\mathbf{G}(\xi + 1/z)\|_2 = \|\mathbf{G}\|_{H^\infty} < 1, \quad (6.23)$$

10 so that \mathbf{H} is strictly bounded real. It follows from the equalities in (6.17) and (6.18)
 11 that $(\mathcal{A}, \mathcal{B}, -\mathcal{C}, \mathcal{D})$ is bounded real balanced, and so $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ is the bounded
 12 real balanced truncation. Invoking [37, Theorem 2] yields that $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1)$ is
 13 minimal, and hence so is (A_ξ, B_ξ, C_ξ) via the relationships in (6.6), establishing
 14 minimality.

15 To establish the strict bounded realness of $(A_\xi, B_\xi, C_\xi, D_\xi)$, again we consider $\xi \in \mathbb{C}_0$
 16 and $\xi \in i\mathbb{R}$ separately. In both cases, let the realisation $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ have
 17 transfer function denoted \mathbf{H}_r^ξ . For $\xi \in \mathbb{C}_0$ we use proof by contraposition; suppose
 18 that $\omega_0 \in \mathbb{R}$ and $u_0 \in \mathbb{C}^m$ with $\|u_0\|_2 = 1$ are such that

$$\|\mathbf{G}_r^\xi(i\omega_0)\|_2 = \|\mathbf{G}_r^\xi(i\omega_0)u_0\|_2 = 1.$$

19 It follows from Lemma 6.3, notably (6.11), that

$$\|\mathbf{H}_r^\xi(p_0)u_0\|_2 = \left\| \mathbf{H}_r^\xi \left(\frac{1}{i\omega_0 - \xi} \right) u_0 \right\|_2 = \|\mathbf{G}_r^\xi(i\omega_0)u_0\|_2 = 1,$$

20 where $p_0 := 1/(i\omega_0 - \xi) \in \partial\mathbb{E}_\xi$.

An elementary sequence of calculations using (6.9) and (6.17), which are relegated
 to Appendix B, shows that

$$\begin{aligned} & I - [\mathbf{H}_r^\xi(p_0)]^* \mathbf{H}_r^\xi(p_0) \\ &= q^2 (\mathcal{B}_2 + \mathcal{A}_{21}(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1)^* \Sigma_2 (\mathcal{B}_2 + \mathcal{A}_{21}(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1) \\ & \quad + (\mathcal{W} - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1)^* (\mathcal{W} - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1), \end{aligned} \quad (6.24)$$

21 where $q := \sqrt{2\operatorname{Re}(\xi)} > 0$. Since $\Sigma_2 > 0$, in light of (6.24), it follows that

$$(\mathcal{B}_2 + \mathcal{A}_{21}(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1)u_0 = 0, \quad (6.25)$$

22 and

$$(\mathcal{W} - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1)u_0 = 0. \quad (6.26)$$

23 Setting

$$z_0 := \begin{pmatrix} (p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1 u_0 \\ 0 \end{pmatrix},$$

and appealing to (6.25), we have that

$$\begin{aligned} \mathcal{A}z_0 + \mathcal{B}u_0 &= \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \begin{pmatrix} (p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1 u_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} u_0 \\ &= \begin{pmatrix} p_0(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1 u_0 \\ 0 \end{pmatrix} \\ &= p_0 z_0. \end{aligned} \tag{6.27}$$

1 Since $\sigma(\mathcal{A}) \subseteq \mathbb{E}_\omega$, $p_0 \notin \sigma(\mathcal{A})$, and so rearranging (6.27) yields

$$z_0 = (p_0 I - \mathcal{A})^{-1} \mathcal{B}u_0.$$

We conclude that

$$\begin{aligned} (\mathcal{W} - \mathcal{K}(p_0 I - \mathcal{A})^{-1} \mathcal{B})u_0 &= \mathcal{W}u_0 - \mathcal{K}z_0 = \begin{pmatrix} \mathcal{K}_1 & \mathcal{K}_2 \end{pmatrix} \begin{pmatrix} (p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1 u_0 \\ 0 \end{pmatrix} \\ &= \mathcal{W}u_0 - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1} \mathcal{B}_1 u_0 \\ &= 0, \end{aligned} \tag{6.28}$$

2 by (6.26). Another elementary series of calculations using (6.9) and (6.17), relegated
3 to Appendix C, shows that

$$I - [\mathbf{H}(p_0)]^* \mathbf{H}(p_0) = (\mathcal{W} - \mathcal{K}(p_0 I - \mathcal{A})^{-1} \mathcal{B})^* (\mathcal{W} - \mathcal{K}(p_0 I - \mathcal{A})^{-1} \mathcal{B}), \tag{6.29}$$

4 which, in conjunction with (6.28), implies that

$$\|\mathbf{H}(p_0)u_0\|_2 = 1.$$

5 Invoking (6.10), we now see that

$$\|\mathbf{G}(i\omega_0)u\|_2 = \|\mathbf{H}(p_0)u_0\|_2 = 1,$$

6 implying that \mathbf{G} is not strictly bounded real. The above proof is easily altered by
7 taking $p_0 = 0$ in the case that

$$\lim_{\substack{\omega \in \mathbb{R} \\ \omega \rightarrow \infty}} \|\mathbf{G}_r^\xi(i\omega)\|_2 = 1,$$

8 as \mathbf{G}_r^ξ is continuous at infinity.

9 It remains to consider $\xi \in i\mathbb{R}$. We first establish that $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ is strictly
10 bounded real. For which purpose, the inequality (6.23) implies that $\|\mathcal{D}\|_2 < 1$, and
11 hence $I - \mathcal{D}^* \mathcal{D}$ is invertible. Since $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ is bounded real balanced, it
12 follows from the Bounded Real Lemma and by construction that Σ_1 and Σ_1^{-1} are
13 solutions of the bounded real algebraic Riccati equation

$$\mathcal{A}_{11}^* \Sigma_1 + \Sigma_1 \mathcal{A}_{11} + \mathcal{C}_1^* \mathcal{C}_1 + (Z \mathcal{B}_1 - \mathcal{C}_1^* \mathcal{D})(I - \mathcal{D}^* \mathcal{D})^{-1} (Z \mathcal{B}_1 - \mathcal{C}_1^* \mathcal{D})^* = 0, \tag{6.30}$$

14 with the property that $\Sigma_1^{-1} > I > \Sigma_1$. For notational convenience, define

$$\mathcal{R} := I - \mathcal{D}^* \mathcal{D} = \mathcal{R}^* > 0, \quad \mathcal{S} = I - \mathcal{D} \mathcal{D}^* = \mathcal{S}^* > 0,$$

15 and

$$\mathcal{A}_E := \mathcal{A}_{11} + \mathcal{B}_1 \mathcal{R}^{-1} (\mathcal{B}_1^* \Sigma_1 - \mathcal{D}^* \mathcal{C}_1).$$

16 In light of [50, Theorem 13.19], it suffices to prove that \mathcal{A}_E is Hurwitz, that is, that
17 Σ_1 is a stabilizing solution of (6.30). Elementary manipulation of (6.30) for both
18 $Z = \Sigma_1$ and $Z = \Sigma_1^{-1}$ shows that

$$\mathcal{A}_E^* \Sigma_1 + \Sigma_1 \mathcal{A}_E + \mathcal{C}_1^* \mathcal{S}^{-1} \mathcal{C}_1 - \Sigma_1 \mathcal{B}_1 \mathcal{R}^{-1} \mathcal{B}_1^* \Sigma_1 = 0, \tag{6.31}$$

1 and

$$\mathcal{A}_E^* \Sigma_1^{-1} + \Sigma_1^{-1} \mathcal{A}_E + \mathcal{C}_1^* \mathcal{S}^{-1} \mathcal{C}_1 + \Pi \mathcal{B}_1 \mathcal{R}^{-1} \mathcal{B}_1^* \Sigma_1 \Pi - \Sigma_1 \mathcal{B}_1 \mathcal{R}^{-1} \mathcal{B}_1^* \Sigma_1 = 0, \quad (6.32)$$

2 hold, where $\Pi = \Sigma_1^{-1} - \Sigma_1 = \Pi^* > 0$. Subtracting (6.31) from (6.32) gives

$$\mathcal{A}_E^* \Pi + \Pi \mathcal{A}_E + \Pi \mathcal{B}_1 \mathcal{R}^{-1} \mathcal{B}_1^* \Pi = 0,$$

3 from which we see that every eigenvalue of \mathcal{A}_E has non-positive real part. Now
4 suppose that $v \in \mathbb{C}^r$ and $\omega \in \mathbb{R}$ are such that $\mathcal{A}_E v = i\omega v$. Forming the inner
5 product

$$\langle [\mathcal{A}_E^* \Pi + \Pi \mathcal{A}_E + \Pi \mathcal{B}_1 \mathcal{R}^{-1} \mathcal{B}_1^* \Pi] v, v \rangle = 0,$$

6 it follows that

$$\mathcal{B}_1^* \Pi v = 0. \quad (6.33)$$

7 Since

$$\langle [\mathcal{A}_E^* \Pi + \Pi \mathcal{A}_E + \Pi \mathcal{B}_1 \mathcal{R}^{-1} \mathcal{B}_1^* \Pi] x, v \rangle = 0 \quad \forall x \in \mathbb{C}^r,$$

we see that

$$\begin{aligned} \langle [\mathcal{A}_E^* \Pi + \Pi \mathcal{A}_E] x, v \rangle = 0 \quad \forall x \in \mathbb{C}^r &\Rightarrow \langle x, [\mathcal{A}_E^* + i\omega I] \Pi v \rangle = 0 \quad \forall x \in \mathbb{C}^r \\ &\Rightarrow \mathcal{A}_E^* \Pi v = -i\omega \Pi v. \end{aligned} \quad (6.34)$$

8 Finally, noting that $(\mathcal{A}_E, \mathcal{B}_1)$ is controllable, as $(\mathcal{A}_{11}, \mathcal{B}_1)$ is, we conclude from (6.33)
9 and (6.34) that $\Pi v = 0$, and so $v = 0$. Hence, \mathcal{A}_E is Hurwitz and so $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$
10 is strictly bounded real. Finally, invoking (6.11) and that $\xi \in i\mathbb{R}$, we estimate that

$$\|\mathbf{G}_r^\xi\|_{H^\infty} = \sup_{z \in \mathbb{C}_0} \|\mathbf{G}_r^\xi(z)\|_2 = \sup_{z \in \mathbb{C}_0} \|\mathbf{H}_r^\xi(1/(z - \xi))\|_2 = \|\mathbf{H}_r^\xi\|_{H^\infty} < 1,$$

11 whence $(A_\xi, B_\xi, C_\xi, D_\xi)$ is strictly bounded real. \square

12 *Proof of Theorem 3.3.* Let (A, B, C, D) denote a minimal, bounded real balanced,
13 and stable, realisation of \mathbf{G} . For K, W, L, X as in (3.1) and (3.2), it follows that
14 the realisation

$$\left(A, \begin{pmatrix} B & L \end{pmatrix}, \begin{pmatrix} C \\ K \end{pmatrix}, \begin{pmatrix} D & X \\ W & 0 \end{pmatrix} \right), \quad (6.35)$$

15 with transfer function \mathbf{J} , is Lyapunov balanced. Let $(A_\xi, B_\xi, C_\xi, D_\xi)$, with transfer
16 function \mathbf{G}_r^ξ , denote the bounded real GSPA of (A, B, C, D) , which is well-defined
17 for all $\xi \in \mathbb{C}_0 \cup i\mathbb{R}$ by Theorem 3.2. By construction, the realisation

$$\left(A_\xi, \begin{pmatrix} B_\xi & L_\xi \end{pmatrix}, \begin{pmatrix} C_\xi \\ K_\xi \end{pmatrix}, \begin{pmatrix} D_\xi & X_\xi \\ W_\xi & 0 \end{pmatrix} \right), \quad (6.36)$$

18 is the GSPA of that in (6.35), where K_ξ, L_ξ, W_ξ and X_ξ are given by (6.16).

19 Letting \mathbf{J}_r^ξ denote the transfer function of (6.36) and invoking Theorem 2.4 yields

$$\|\mathbf{J} - \mathbf{J}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^n \sigma_j, \quad (6.37)$$

20 where $(\sigma_j)_{j=1}^n$ are the Hankel singular values of \mathbf{J} , which are equal to the bounded
21 real singular values of \mathbf{G} . Combining (6.37) with the easily established estimate

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq \|\mathbf{J} - \mathbf{J}_r^\xi\|_{H^\infty},$$

22 gives (3.3), as required. The function \mathbf{G}_r^ξ has the properties claimed.

23 The final claim follows from statement (iii) of Theorem 3.2. \square

1 *Proof of Proposition 3.4:* The proof builds on that of Theorem 3.3.

2 For statement (i), define $\mathbf{R} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$ and $\mathbf{S} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times p})$ by the
3 realisations

$$(A, B, K, W) \quad \text{and} \quad (A, L, C, X),$$

4 respectively. In light of (3.1) and (3.2), it follows from statements (ii) and (iii) of
5 Lemma 6.1 that \mathbf{R} and \mathbf{S} are spectral factors of $I - \mathbf{G}^* \mathbf{G}$ and $I - \mathbf{G} \mathbf{G}^*$, respectively,
6 as required.

7 For statement (ii), let $\xi \in i\mathbb{R}$, and let $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$ and $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times p})$
8 be defined by the realisations

$$(A_\xi, B_\xi, K_\xi, W_\xi) \quad \text{and} \quad (A_\xi, L_\xi, C_\xi, X_\xi), \quad (6.38)$$

9 respectively, where K_ξ, L_ξ, W_ξ and X_ξ are given by (6.16). Appealing to (6.14),
10 (6.15), and invoking statements (ii) and (iii) of Lemma 6.1, it follows that \mathbf{R}_r^ξ and
11 \mathbf{S}_r^ξ are spectral factors of \mathbf{G}_r^ξ in the sense of (3.4), as required. By their definitions
12 in.

13 The error bound (3.5) follows by combining (6.37) with the identity

$$\begin{pmatrix} \mathbf{G} - \mathbf{G}_r^\xi & \mathbf{S} - \mathbf{S}_r^\xi \\ \mathbf{R} - \mathbf{R}_r^\xi & \# \end{pmatrix} = \mathbf{J} - \mathbf{J}_r^\xi,$$

14 (which follows by construction) where $\#$ denotes an entry we are not concerned with.
15 The error bounds (3.6) are a straightforward consequence of (3.5).

16 The interpolation equalities (3.7) hold owing to the definition (6.16) of the realisa-
17 tion (6.38) (compare with (2.5)).

18 For statement (iii), we define $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$ and $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times p})$ as
19 above, which, as with the proof of statement (ii), satisfy properties (3.5)–(3.7).
20 Appealing to (6.14), an application of statement (ii) of Lemma 6.1, the function
21 $\mathbf{U}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{2m \times m})$ with realisation

$$\left(A_\xi, B_\xi, \begin{pmatrix} K_\xi \\ q\sqrt{\Sigma_2}\phi A_{21} \end{pmatrix}, \begin{pmatrix} W_\xi \\ q\sqrt{\Sigma_2}\phi B_2 \end{pmatrix} \right),$$

22 where $q := \sqrt{2\operatorname{Re}(\xi)} > 0$ and $\phi = (\xi I - A_{22})^{-1}$, is a spectral factor of $I - (\mathbf{G}_r^\xi)^* \mathbf{G}_r^\xi$.
23 A straightforward calculation shows that

$$(\mathbf{U}_r^\xi)^* \mathbf{U}_r^\xi \geq (\mathbf{R}_r^\xi)^* \mathbf{R}_r^\xi \quad \text{on } i\mathbb{R},$$

24 establishing the first inequality in (3.8). The dual case is proven similarly, us-
25 ing (6.15), and invoking statement (iii) of Lemma 6.1 with $\mathbf{V}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p \times 2p})$
26 defined by the realisation

$$(A_\xi, (L_\xi \quad qA_{12}\phi\sqrt{\Sigma_2}), C_\xi, (X_\xi \quad qC_2\phi\sqrt{\Sigma_2})).$$

27 □

28 6.2. The positive real generalised singular perturbation approximation.

29 The proof of the next lemma is very similar to that of Lemma 6.1, and is thus
30 omitted. We have also omitted the corresponding statements pertaining to the dual
31 positive real equations as, although they do hold, we shall not require them.

1 **Lemma 6.4.** *If (A, B, C, D) with transfer function \mathbf{G} and $\Sigma \geq 0$ are such that*

$$\begin{aligned} A^* \Sigma + \Sigma A &= -K^* K - P^* P, \\ \Sigma B - C^* &= -K^* W - P^* Q, \\ D^* + D &= W^* W + Q^* Q, \end{aligned}$$

2 *for some appropriately sized K, P, Q and W , then the following statements hold.*

- 3 (i) (A, B, C, D) *is positive real.*
 4 (ii) \mathbf{R} *with realisation $(A, B, \begin{bmatrix} K \\ P \end{bmatrix}, \begin{bmatrix} W \\ Q \end{bmatrix})$ is a spectral factor for $\mathbf{G}^* + \mathbf{G}$ in the*
 5 *sense that*

$$(\mathbf{G}(s))^* + \mathbf{G}(s) = (\mathbf{R}(s))^* \mathbf{R}(s) \quad \forall s \in i\mathbb{R} \setminus \Delta,$$

6 *where Δ denotes the set of poles of \mathbf{G} .*

7 We shall employ the so-called Cayley Transform $\mathcal{S} : H(\mathbb{C}_0, \mathbb{C}^{m \times m}) \supseteq D(\mathcal{S}) \rightarrow$
 8 $H(\mathbb{C}_0, \mathbb{C}^{m \times m})$, which is given by

$$\mathcal{S}(\mathbf{G})(s) = (I - \mathbf{G}(s))(I + \mathbf{G}(s))^{-1} \quad s \in \mathbb{C}_0.$$

9 Here $D(\mathcal{S})$ contains all $\mathbf{G} \in H(\mathbb{C}_0, \mathbb{C}^{m \times m})$ where the above formula makes sense (at
 10 least) for all $s \in \mathbb{C}_0$. Further, it is well-known (see, instance, [19, Lemma 7.1.8]) that
 11 if \mathbf{G} is positive real, then $\mathbf{G} \in D(\mathcal{S})$ and $\mathcal{S}(\mathbf{G})$ is bounded real, and so in particular,
 12 belongs to $H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$. It is evident that the Cayley transform maps rational
 13 functions to rational functions.

14 If (A, B, C, D) is a minimal realisation of $\mathbf{G} \in D(\mathcal{S})$, then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ given by

$$\left. \begin{aligned} \tilde{A} &:= A - B(I + D)^{-1}C & \tilde{B} &:= \sqrt{2}B(I + D)^{-1} \\ \tilde{C} &:= -\sqrt{2}(I + D)^{-1}C & \tilde{D} &:= (I - D)(I + D)^{-1} \end{aligned} \right\}, \quad (6.39)$$

15 is well-defined and a minimal realisation of $\mathcal{S}(\mathbf{G})$. Since $\mathcal{S} : D(\mathcal{S}) \rightarrow D(\mathcal{S})$ and
 16 $\mathcal{S}^2 = \text{id}$, the identity function, meaning that \mathcal{S} is self-inverse, it follows that

$$(\tilde{\tilde{A}}, \tilde{\tilde{B}}, \tilde{\tilde{C}}, \tilde{\tilde{D}}) \text{ is well-defined and } (\tilde{\tilde{A}}, \tilde{\tilde{B}}, \tilde{\tilde{C}}, \tilde{\tilde{D}}) = (A, B, C, D).$$

17 The next lemma shows that the following diagram

$$\left. \begin{array}{ccc} (A, B, C, D) & \xrightarrow{\text{GSPA}} & (A_\xi, B_\xi, C_\xi, D_\xi) \\ \uparrow \text{Cayley} & & \uparrow \text{Cayley} \\ (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) & \xrightarrow{\text{GSPA}} & ((\tilde{A})_\xi, (\tilde{B})_\xi, (\tilde{C})_\xi, (\tilde{D})_\xi) \end{array} \right\} \quad (6.40)$$

18 commutes. The proof is a tedious series of elementary calculations, and is relegated
 19 to Appendix D.

20 **Lemma 6.5.** *Given $\xi \in \mathbb{C}$ with $\text{Re}(\xi) \geq 0$ and (A, B, C, D) , assume that each of*
 21 *the quadruples in (6.40) are well-defined. Then*

$$((\tilde{\tilde{A}})_\xi, (\tilde{\tilde{B}})_\xi, (\tilde{\tilde{C}})_\xi, (\tilde{\tilde{D}})_\xi) = ((\tilde{A})_\xi, (\tilde{B})_\xi, (\tilde{C})_\xi, (\tilde{D})_\xi),$$

22 *and so the diagram (6.40) commutes.*

23 *Proof of Theorem 4.2.* Let $\xi \in \mathbb{C}$ with $\text{Re}(\xi) \geq 0$. An application of [40, Theorem
 24 3.2] to the first two equations in (4.2) and (4.3) shows that A_{22} is Hurwitz, so that

- 1 $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well-defined. Elementary calculations using the definitions of
 2 $(A_\xi, B_\xi, C_\xi, D_\xi)$ in (2.5) and the equalities (4.2) considered block wise show that

$$\left. \begin{aligned} A_\xi^* \Sigma_1 + \Sigma_1 A_\xi &= -K_\xi^* K_\xi - 2\operatorname{Re}(\xi) A_{21}^* \phi^* \Sigma_2 \phi A_{21} \\ \Sigma_1 B_\xi - C_\xi^* &= -K_\xi^* W_\xi - 2\operatorname{Re}(\xi) A_{21}^* \phi^* \Sigma_2 \phi B_2 \\ D_\xi^* + D_\xi &= W_\xi^* W_\xi + 2\operatorname{Re}(\xi) B_2^* \phi^* \Sigma_2 \phi B_2 \end{aligned} \right\}, \quad (6.41)$$

- 3 and

$$\left. \begin{aligned} A_\xi \Sigma_1 + \Sigma_1 A_\xi^* &= -L_\xi L_\xi^* - 2\operatorname{Re}(\xi) A_{12} \phi \Sigma_2 \phi^* A_{12}^* \\ \Sigma_1 C_\xi^* - B_\xi &= -L_\xi X_\xi^* - 2\operatorname{Re}(\xi) A_{12} \phi \Sigma_2 \phi^* C_2^* \\ D_\xi + D_\xi^* &= X_\xi X_\xi^* + 2\operatorname{Re}(\xi) C_2 \phi \Sigma_2 \phi^* C_2^* \end{aligned} \right\}, \quad (6.42)$$

- 4 where $\phi = (\xi I - A_{22})^{-1}$ and $K_\xi, W_\xi, L_\xi, X_\xi$ are given by (6.16).

- 5 In light of (6.41), an application of statement (i) of Lemma 6.4 yields that $(A_\xi, B_\xi, C_\xi, D_\xi)$
 6 is positive real. Evidently, if $\xi \in i\mathbb{R}$, then the resulting simplification of (6.41)
 7 and (6.42) implies that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is positive real balanced, completing the
 8 proof of statement (i).

- 9 The proof that A_ξ is Hurwitz when $\xi \in \mathbb{C}_0$ is the same as that in the proof of
 10 Theorem 3.2, only using the first equation in (6.41), instead of (6.14). The details
 11 are therefore omitted.

- 12 Next, define $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ as in (6.5) and note that $\mathcal{A} = (A - \xi I)^{-1}$ is Hurwitz by
 13 statement (3) of Lemma 6.2. Calculations starting from (4.2) and (4.3) respectively
 14 show that

$$\left. \begin{aligned} \mathcal{A}^* \Sigma + \Sigma \mathcal{A} &= -\mathcal{K}^* \mathcal{K} - 2\operatorname{Re}(\xi) \mathcal{A}^* \Sigma \mathcal{A} \\ \Sigma \mathcal{B} - (-\mathcal{C})^* &= \mathcal{K}^* \mathcal{W} - 2\operatorname{Re}(\xi) \mathcal{A}^* \Sigma \mathcal{B} \\ \mathcal{D}^* + \mathcal{D} &= \mathcal{W}^* \mathcal{W} + 2\operatorname{Re}(\xi) \mathcal{B}^* \Sigma \mathcal{B} \end{aligned} \right\}, \quad (6.43)$$

- 15 and

$$\left. \begin{aligned} \mathcal{A} \Sigma + \Sigma \mathcal{A}^* &= -\mathcal{L} \mathcal{L}^* - 2\operatorname{Re}(\xi) \mathcal{A} \Sigma \mathcal{A}^* \\ \Sigma(-\mathcal{C})^* - \mathcal{B} &= -\mathcal{L} \mathcal{X}^* - 2\operatorname{Re}(\xi) \mathcal{A} \Sigma(-\mathcal{C})^* \\ \mathcal{D} + \mathcal{D}^* &= \mathcal{X} \mathcal{X}^* + 2\operatorname{Re}(\xi) \mathcal{C} \Sigma \mathcal{C}^* \end{aligned} \right\}, \quad (6.44)$$

- 16 where $\mathcal{K}, \mathcal{W}, \mathcal{L}$ and \mathcal{X} are given by (6.19).

- 17 When $\xi \in i\mathbb{R}$, then a consequence of the first equations in (6.43) and (6.44) is
 18 that the realisation $(\mathcal{A}, \mathcal{L}, \mathcal{K})$ is Lyapunov balanced. Thus \mathcal{A}_{11} is Hurwitz by [40,
 19 Theorem 3.2], again invoking the assumption that the singular values are simple
 20 implies that the spectra of Σ_1 and Σ_2 are disjoint. Statement (2) of Lemma 6.2
 21 yields that $\xi \notin \sigma(A_\xi)$. Consequently, $A_\xi - \xi I$ is invertible, and thus from (6.6) we
 22 see that $\mathcal{A}_{11} = (A_\xi - \xi I)^{-1}$. It is now routine to verify that A_ξ is Hurwitz, since
 23 \mathcal{A}_{11} is, and $\xi \in i\mathbb{R}$. We have proven statement (ii).

- 24 To prove statement (iii), assume that (A, B, C, D) is strongly positive real, so that
 25 $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is well-defined and strictly bounded real. Further, \tilde{A} is Hurwitz, since
 26 the realisation $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is minimal, and the transfer function is strictly bounded
 27 real (and hence belongs to H^∞).

- 28 As (A, B, C, D) is assumed positive real balanced, it follows that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is
 29 bounded real balanced (by [37, Lemma 5]). Invoking statement (iii) of Theorem 3.2,
 30 it follows that

$$((\tilde{A})_\xi, (\tilde{B})_\xi, (\tilde{C})_\xi, (\tilde{D})_\xi),$$

1 is minimal and strictly bounded real, and so is

$$((\widetilde{A_\xi}), (\widetilde{B_\xi}), (\widetilde{C_\xi}), (\widetilde{D_\xi})),$$

2 by Lemma 6.5. Since the Cayley transform is self-inverse, preserves minimality and
3 maps strictly bounded real systems to strongly positive real systems [19, Lemma
4 7.1.8, p.159], it follows that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and strongly positive real,
5 proving statement (iii). \square

6 *Proof of Theorem 4.3.* Let (A, B, C, D) denote a minimal, positive real balanced
7 realisation of \mathbf{G} and $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ which is not a pole of \mathbf{G} . Therefore, ξ
8 is not an eigenvalue of A , as (A, B, C) is minimal. Arguing as in the proof of [40,
9 Theorem 3.2] from the first equations in (4.2) and (4.3) shows that $\xi \notin \sigma(A_{22})$, and
10 so $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well defined.

11 Let \mathbf{G}_r^ξ and \mathbf{H} be defined by the realisations

$$(A_\xi, B_\xi, C_\xi, D_\xi) \quad \text{and} \quad (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}),$$

12 respectively. In light of (6.41), an application of statement (i) of Lemma 6.4
13 yields that \mathbf{G}_r^ξ is positive real. Therefore, $\mathbf{G}_r^\xi \in D(\mathcal{S})$, in particular meaning that
14 $((\widetilde{A_\xi}), (\widetilde{B_\xi}), (\widetilde{C_\xi}), (\widetilde{D_\xi}))$ is well-defined. Next, note that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is minimal,
15 stable, bounded real, and bounded real balanced, whence \tilde{A}_{22} is Hurwitz and so
16 $((\tilde{A})_\xi, (\tilde{B})_\xi, (\tilde{C})_\xi, (\tilde{D})_\xi)$ is well-defined; we denote its transfer function by \mathbf{H}_r^ξ .

17 A consequence of Lemma 6.5 is that $\mathcal{S}(\mathbf{G}_r^\xi) = \mathbf{H}_r^\xi$. An application of Theorem 3.3
18 shows that

$$\|\mathbf{H} - \mathbf{H}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^n \sigma_j,$$

19 since the positive real singular values of \mathbf{G} are precisely the bounded real singular
20 values of \mathbf{H} , see [23, Corollary 9.6]. The remainder of the proof of (4.4) follows using
21 the arguments given in [19, Theorem 7.2.12] or [22, Theorem 1.2]. The bound (4.5)
22 follows from (4.4) and the equivalence of the gap metric restricted to bounded,
23 linear operators and the operator norm, see [19, Corollary 3.6.9].

24 If $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$, then, in addition to its other properties, the realisation
25 (A, B, C, D) may be chosen to be stable. It follows from statement (ii) of Theo-
26 rem 4.2 that A_ξ is Hurwitz and so $\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m \times m})$ as well. If \mathbf{G} is strongly
27 positive real, then, by construction of \mathbf{G}_r^ξ , statement (iii) of Theorem 4.2 implies
28 that \mathbf{G}_r^ξ is strongly positive real as well. \square

29 *Proof of Proposition 4.4.* (i) Since \mathbf{G} is positive real, $\mathbf{G} \in D(\mathcal{S})$ and $\mathbf{H} := \mathcal{S}(\mathbf{G})$ is
30 bounded real. Applying statement (i) of Proposition 3.4 to $\mathbf{H} \in H^\infty$ yields $\mathbf{T} \in H^\infty$
31 such that

$$I - \mathbf{H}^* \mathbf{H} = \mathbf{T}^* \mathbf{T} \quad \text{on } i\mathbb{R}. \quad (6.45)$$

Since $\mathbf{H} \in D(\mathcal{S})$ and \mathcal{S} is self-inverse, we have that $\mathbf{G} = \mathcal{S}(\mathbf{H})$ and a straightforward
calculation invoking (6.45) shows that

$$\begin{aligned} \mathbf{G} + \mathbf{G}^* &= \mathcal{S}(\mathbf{H}) + [\mathcal{S}(\mathbf{H})]^* = (I - \mathbf{H})(I + \mathbf{H})^{-1} + [(I - \mathbf{H})(I + \mathbf{H})^{-1}]^* \\ &= 2(I + \mathbf{H})^{-*} [I - \mathbf{H}^* \mathbf{H}] (I + \mathbf{H})^{-1} \\ &= (\mathbf{R})^* \mathbf{R} \quad \forall s \in i\mathbb{R} \setminus \Delta, \end{aligned}$$

1 where $\mathbf{R} := \sqrt{2}\mathbf{T}(I + \mathbf{H})^{-1}$, which is evidently rational. Moreover, upon calculating

$$(I + \mathbf{H})^{-1} = (I + \mathcal{S}(\mathbf{G}))^{-1} = \frac{1}{2}(I + \mathbf{G}),$$

2 it follows that \mathbf{R} is proper.

3 (ii) The proof mimics that of statement (i), only replacing \mathbf{G} by \mathbf{G}_r^ξ from Theo-
4 rem 4.3 and $\mathbf{H}_r^\xi := \mathcal{S}(\mathbf{G}_r^\xi)$. Then (6.45) becomes

$$I - (\mathbf{H}_r^\xi)^* \mathbf{H}_r^\xi = (\mathbf{T}_r^\xi)^* \mathbf{T}_r^\xi \quad \text{on } i\mathbb{R}, \quad (6.46)$$

5 for some $\mathbf{T}_r^\xi \in H^\infty$. The desired proper, rational spectral factor \mathbf{R}_r^ξ is given by
6 $\mathbf{R}_r^\xi := \sqrt{2}\mathbf{T}_r^\xi(I + \mathbf{H}_r^\xi)^{-1} = (\sqrt{2}/2)\mathbf{T}(I + \mathbf{G}_r^\xi)$. Note that since $\mathbf{G}(\xi) = \mathbf{G}_r^\xi$, we have
7 that

$$\mathbf{H}(\xi) = (I - \mathbf{G}(\xi))(I + \mathbf{G}(\xi))^{-1} = (I - \mathbf{G}_r^\xi(\xi))(I + \mathbf{G}_r^\xi(\xi))^{-1} = \mathbf{H}_r^\xi(\xi).$$

8 Therefore, we verify that

$$\mathbf{R}(\xi) = \sqrt{2}\mathbf{T}(\xi)(I + \mathbf{H}(\xi))^{-1} = \sqrt{2}\mathbf{T}_r^\xi(\xi)(I + \mathbf{H}_r^\xi(\xi))^{-1} = \mathbf{R}_r^\xi(\xi),$$

9 where we have used $\mathbf{T}(\xi) = \mathbf{T}_r^\xi(\xi)$, which follows from (3.7).

10 By Theorem 4.3, if $\mathbf{G} \in H^\infty$, then $\mathbf{G}_r^\xi \in H^\infty$ as well, whence so are $\mathbf{R}, \mathbf{R}_r^\xi$.

Finally, using the definitions of \mathbf{R} and \mathbf{R}_r^ξ , we estimate

$$\begin{aligned} \frac{1}{\sqrt{2}} \|\mathbf{R} - \mathbf{R}_r^\xi\|_{H^\infty} &= \|\mathbf{T}(I + \mathbf{H})^{-1} - \mathbf{T}_r^\xi(I + \mathbf{H}_r^\xi)^{-1}\|_{H^\infty} \\ &\leq \|\mathbf{T}((I + \mathbf{H})^{-1} - (I + \mathbf{H}_r^\xi)^{-1})\|_{H^\infty} + \|(\mathbf{T} - \mathbf{T}_r^\xi)(I + \mathbf{H}_r^\xi)^{-1}\|_{H^\infty} \\ &\leq \frac{1}{2} \|\mathbf{T}\|_{H^\infty} \|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} + \|\mathbf{T} - \mathbf{T}_r^\xi\|_{H^\infty} \|(I + \mathbf{H}_r^\xi)^{-1}\|_{H^\infty} \\ &\leq (a \|\mathbf{T}\|_{H^\infty} + 2 \|(I + \mathbf{H}_r^\xi)^{-1}\|_{H^\infty}) \sum_{j=r+1}^n \sigma_j, \end{aligned} \quad (6.47)$$

where we have invoked (4.5) and (3.6) in the final inequality above. Using expressions for \mathbf{T} and $(I + \mathbf{H}_r^\xi)^{-1}$ yields that

$$\|\mathbf{R} - \mathbf{R}_r^\xi\|_{H^\infty} \leq \left(2a \|\mathbf{R}(I + \mathbf{G})^{-1}\|_{H^\infty} + \sqrt{2} \|I + \mathbf{G}_r^\xi\|_{H^\infty} \right) \sum_{j=r+1}^n \sigma_j. \quad (6.48)$$

If in (6.47) we add and subtract $\mathbf{T}_r^\xi(I + \mathbf{H})^{-1}$ (instead of $\mathbf{T}(I + \mathbf{H}_r^\xi)^{-1}$) and perform the analogous steps, *mutatis mutandis*, we arrive at the bound

$$\|\mathbf{R} - \mathbf{R}_r^\xi\|_{H^\infty} \leq \left(2a \|\mathbf{R}_r^\xi(I + \mathbf{G}_r^\xi)^{-1}\|_{H^\infty} + \sqrt{2} \|I + \mathbf{G}\|_{H^\infty} \right) \sum_{j=r+1}^n \sigma_j. \quad (6.49)$$

11 Combining (6.48) and (6.49) gives the required bound. \square

12 **Appendix A. Proofs of Theorems 2.3 and 2.4.** We need the following lemma.

13 **Lemma A.1.** *Given $\xi \in \mathbb{C}_0$, suppose that $(A, B, -C, D)$ with transfer function \mathbf{H}*
14 *satisfies*

$$A\Sigma + \Sigma A^* + BB^* \leq -2\operatorname{Re}(\xi)A\Sigma A^*, \quad (A.1)$$

15 and

$$A^*\Sigma + \Sigma A + C^*C \leq -2\operatorname{Re}(\xi)A^*\Sigma A. \quad (A.2)$$

- 1 Further assume that $\Sigma = \Sigma^* > 0$ has simple eigenvalues $(\sigma_j)_{j=1}^n$, ordered according
 2 to (2.3), and that for each $k \in \{r, \dots, n\}$ the truncation $A_{11}^{(k)} \in \mathbb{C}^{k \times k}$ satisfies

$$\sigma(A_{11}^{(k)}) \subseteq \mathbb{E}_\xi, \quad (\text{A.3})$$

- 3 where $A_{11}^{(r)} = A_{11}$ and $A_{11}^{(n)} = A$. Let \mathbf{H}_r have realisation $(A_{11}, B_1, -C_1, D)$. Then

$$\|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2 \leq 2 \sum_{j=r+1}^n \sigma_j \quad \forall s \in \partial\mathbb{E}_\xi. \quad (\text{A.4})$$

- 4 If $\xi \in i\mathbb{R}$, (A.1) and (A.2) hold, and (A.3) is replaced by

$$A_{11}^{(k)} \text{ is Hurwitz for all } k \in \{r, \dots, n\},$$

- 5 then

$$\|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2 \leq 2 \sum_{j=r+1}^n \sigma_j \quad \forall s \in i\mathbb{R}. \quad (\text{A.5})$$

- 6 *Proof.* First let $\xi \in \mathbb{C}_0$. For $s \in \partial\mathbb{E}_\xi$, let

$$\begin{aligned} A_s &:= A_{22} + A_{21}(sI - A_{11})^{-1}A_{12} \\ B_s &:= B_2 + A_{21}(sI - A_{11})^{-1}B_1, \\ C_s &:= C_2 + C_1(sI - A_{11})^{-1}A_{12} \end{aligned}$$

- 7 which are well-defined by assumption (A.3).

Block wise inspection of the two inequalities (A.1) and (A.2) yields the relationships:

$$\begin{aligned} A_{11}\Sigma_1 + \Sigma_1 A_{11}^* + B_1 B_1^* &\leq -2\text{Re}(\xi) (A_{11}\Sigma_1 A_{11}^* + A_{12}\Sigma_2 A_{12}^*), \\ A_{12}\Sigma_2 + \Sigma_1 A_{21}^* + B_1 B_2^* &\leq -2\text{Re}(\xi) (A_{11}\Sigma_1 A_{21}^* + A_{12}\Sigma_2 A_{22}^*), \\ A_{22}\Sigma_2 + \Sigma_2 A_{22}^* + B_2 B_2^* &\leq -2\text{Re}(\xi) (A_{21}\Sigma_1 A_{21}^* + A_{22}\Sigma_2 A_{22}^*), \end{aligned} \quad (\text{A.6})$$

and

$$\begin{aligned} A_{11}^*\Sigma_1 + \Sigma_1 A_{11} + C_1^* C_1 &\leq -2\text{Re}(\xi) (A_{11}^*\Sigma_1 A_{11} + A_{21}^*\Sigma_2 A_{21}), \\ A_{21}^*\Sigma_2 + \Sigma_1 A_{12} + C_1^* C_2 &\leq -2\text{Re}(\xi) (A_{11}^*\Sigma_1 A_{12} + A_{21}^*\Sigma_2 A_{22}), \\ A_{22}^*\Sigma_2 + \Sigma_2 A_{22} + C_2^* C_2 &\leq -2\text{Re}(\xi) (A_{12}^*\Sigma_1 A_{12} + A_{22}^*\Sigma_2 A_{22}). \end{aligned} \quad (\text{A.7})$$

- 8 An elementary sequence of calculations, using the definitions of A_s , B_s and C_s and
 9 the above inequalities, gives

$$A_s \Sigma_2 + \Sigma_2 A_s^* + B_s B_s^* \leq -2\text{Re}(\xi) A_s \Sigma_2 A_s^*, \quad (\text{A.8})$$

- 10 and

$$A_s^* \Sigma_2 + \Sigma_2 A_s + C_s^* C_s \leq -2\text{Re}(\xi) A_s^* \Sigma_2 A_s. \quad (\text{A.9})$$

We claim that for all $s \in \partial\mathbb{E}_\xi$, $s \notin \sigma(A_s)$ so that $sI - A_s$ is invertible. To establish the claim, if $v \in \mathbb{C}^{n-r}$ is such that $A_s v = sv$, then

$$\begin{aligned} Az &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} (sI - A_{11})^{-1} A_{12} v \\ v \end{pmatrix} = \begin{pmatrix} A_{11}(sI - A_{11})^{-1} A_{12} v + A_{12} v \\ A_s v \end{pmatrix} \\ &= s \begin{pmatrix} (sI - A_{11})^{-1} A_{12} v \\ v \end{pmatrix} = sz. \end{aligned} \quad (\text{A.10})$$

- 11 Since $s \notin \sigma(A)$ (indeed, $\sigma(A) \subseteq \mathbb{E}_\xi$), it follows from (A.10) that $z = 0$ and thus
 12 $v = 0$, proving that $s \notin \sigma(A_s)$.

- 1 Moreover, since $\|C_s v\|^2 \geq 0$ for all $v \in \mathbb{C}^{n-r}$, by considering any eigenvalue λ of A_s
 2 with corresponding eigenvector v and the inequality

$$\langle (A_s^* \Sigma_2 + \Sigma_2 A_s + C_s^* C_s) v, v \rangle \leq -2\operatorname{Re}(\xi) \langle A_s^* \Sigma_2 A_s v, v \rangle,$$

- 3 it follows that

$$2\operatorname{Re}(\lambda) \langle \Sigma_2 v, v \rangle \leq 2\operatorname{Re}(\lambda) \langle \Sigma_2 v, v \rangle + \|C_s v\|^2 \leq -2\operatorname{Re}(\xi) |\lambda|^2 \langle \Sigma_2 v, v \rangle.$$

- 4 Hence,

$$\sigma(A_s) \subseteq \mathbb{E}_\xi \cup \partial \mathbb{E}_\xi, \quad (\text{A.11})$$

- 5 see (6.9). The arguments which follow are, in part, in the spirit of those used
 6 in [10] — deriving the H^∞ error bound for Lyapunov balanced truncation. Setting
 7 $\Delta = \Delta(s) := sI - A_s$, straightforward calculations show that

$$\mathbf{H}(s) - \mathbf{H}_r(s) = C_s \Delta^{-1} B_s \quad \forall s \in \partial \mathbb{E}_\xi,$$

where we have used that $s \notin \sigma(A_s)$, and so

$$\begin{aligned} \|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2^2 &= \lambda_m(C_s \Delta^{-1} B_s (C_s \Delta^{-1} B_s)^*) = \lambda_m(C_s \Delta^{-1} B_s B_s^* \Delta^{-*} C_s^*) \\ &= \lambda_m(\Delta^{-1} B_s B_s^* \Delta^{-*} C_s^* C_s) \quad \forall s \in \partial \mathbb{E}_\xi. \end{aligned} \quad (\text{A.12})$$

- 8 Here we have used that for square matrices M, N and $\lambda \neq 0$, $\lambda \in \sigma(MN)$ if, and
 9 only if, $\lambda \in \sigma(NM)$, and

$$\|M\|_2^2 = \lambda_m(M^* M) =: \max \{ \lambda : \lambda \in \sigma(M^* M) \},$$

- 10 that is, the 2-norm of M is equal to the non-negative squareroot of the largest
 11 eigenvalue of $M^* M$.

For notational convenience in the following arguments set $\zeta = \operatorname{Re}(\xi) > 0$. Rearranging (A.8) yields that

$$B_s B_s^* \leq -(2\zeta A_s \Sigma_2 A_s^* + A_s \Sigma_2 + \Sigma_2 A_s^*),$$

whence

$$\begin{aligned} \Delta^{-1} B_s B_s^* \Delta^{-*} &\leq -(sI - A_s)^{-1} [2\zeta A_s \Sigma_2 A_s^* + A_s \Sigma_2 + \Sigma_2 A_s^*] (sI - A_s)^{-*}, \\ &= -2\zeta ((sI - A_s) - sI) \Sigma_2 ((sI - A_s) - sI)^* \\ &\quad + (sI - A_s) \Sigma_2 + \Sigma_2 (sI - A_s)^* - 2\operatorname{Re}(s) \Sigma_2, \\ &= -2\zeta \Sigma_2 + p \Delta^{-1} \Sigma_2 + \bar{p} \Sigma_2 \Delta^{-*}, \end{aligned} \quad (\text{A.13})$$

where $p := 1 + 2\zeta s$ and we have used (6.9). Similarly, from (A.9), we see that

$$\begin{aligned} C_s^* C_s &\leq -2(\zeta A_s^* \Sigma_2 A_s + A_s^* \Sigma_2 + \Sigma_2 A_s) \\ &= -2\zeta ((sI - A_s) - sI)^* \Sigma_2 ((sI - A_s) - sI) + (sI - A_s)^* \Sigma_2 + \Sigma_2 (sI - A_s) \\ &\quad - 2\operatorname{Re}(s) \Sigma_2, \\ &= -2\zeta \Delta^* \Sigma_2 \Delta + \bar{p} \Sigma_2 \Delta + p \Delta^* \Sigma_2, \end{aligned} \quad (\text{A.14})$$

where again we have used (6.9). Combining (A.13) and (A.14) gives

$$\begin{aligned} &\lambda_m(\Delta^{-1} B_s B_s^* \Delta^{-*} C_s^* C_s) \\ &\leq \lambda_m((-2\zeta \Sigma_2 + p \Delta^{-1} \Sigma_2 + \bar{p} \Sigma_2 \Delta^{-*})(-2\zeta \Delta^* \Sigma_2 \Delta + \bar{p} \Sigma_2 \Delta + p \Delta^* \Sigma_2)) \\ &= \lambda_m((-2\zeta \Delta \Sigma_2 \Delta^* + p \Sigma_2 \Delta^* + \bar{p} \Delta \Sigma_2)(-2\zeta \Sigma_2 + \bar{p} \Delta^{-*} \Sigma_2 + p \Sigma_2 \Delta^{-1})). \end{aligned}$$

Now assume that just one singular value is omitted in the reduced order system, so that $\Sigma_2 = \sigma_n I$. Invoking the assumption that the singular values are simple, it follows that the reduced order system has a scalar state. Then

$$\begin{aligned} & \lambda_m(\Delta^{-1} B_s B_s^* \Delta^{-*} C_s^* C_s) \\ & \leq \sigma_n^2 (-2\zeta \Delta \Delta^* + p \Delta^* + \bar{p} \Delta) (-2\zeta + \bar{p} \Delta^{-*} + p \Delta^{-1}) \\ & = \sigma_n^2 (4\zeta^2 \Delta \Delta^* - 4\zeta p \Delta^* - 4\zeta \bar{p} \Delta + |p|^2 + \bar{p}^2 \Delta \Delta^{-*} + p^2 \Delta^* \Delta^{-1} + |\bar{p}|^2) \\ & = \sigma_n^2 ((1 + \bar{p}^2 \Delta \Delta^{-*})(1 + p^2 \Delta^* \Delta^{-1}) + 4[(\zeta \Delta^* - \bar{p})(\zeta \Delta - p) - 1]), \end{aligned} \quad (\text{A.15})$$

1 where we have used that $|p| = |\bar{p}| = 1$ and that Δ and $\Delta^* = \bar{\Delta}$ are scalar quantities.

We investigate the second term in (A.15) and estimate that

$$\begin{aligned} (\zeta \Delta^* - \bar{p})(\zeta \Delta - p) &= |\zeta \Delta - p|^2 = |\zeta(sI - A_s) - (1 + 2\zeta s)|^2 \\ &= |(-1 - \zeta s) - \zeta A_s|^2 \leq 1, \end{aligned}$$

by geometric considerations and in light of (A.11). Thus the second term in (A.15) is non-positive, and so

$$\lambda_m(\Delta^{-1} B_s B_s^* \Delta^{-*} C_s^* C_s) \leq \sigma_n^2 (1 + \bar{p}^2 \Delta \Delta^{-*})(1 + p^2 \Delta^* \Delta^{-1}) \quad \forall s \in \partial \mathbb{E}_\xi.$$

2 Writing $f(s) = \bar{p}^2 \Delta(s) \Delta^{-*}(s)$, it follows that

$$|f(s)| = \left| \bar{p}^2 \Delta(s) / \overline{\Delta(s)} \right| = 1 \quad \forall s \in \partial \mathbb{E}_\xi,$$

3 therefore

$$\lambda_m(\Delta^{-1} B_s B_s^* \Delta^{-*} C_s^* C_s) \leq \sigma_n^2 |1 + f(s)|^2 \leq \sigma_n^2 (1 + |f(s)|)^2 = 4\sigma_n^2,$$

4 which, when combined with (A.12), proves the one-step bound

$$\|\mathbf{H}_n(s) - \mathbf{H}_{n-1}(s)\|_2 \leq 2\sigma_n \quad \forall s \in \partial \mathbb{E}_\xi,$$

5 where \mathbf{H}_k for $k \in \{1, 2, \dots, n\}$ denotes the reduced order system with k singular
6 values retained so that, in particular, $\mathbf{H}_n = \mathbf{H}$. To establish the intermediate
7 one-step bounds

$$\|\mathbf{H}_j(s) - \mathbf{H}_{j-1}(s)\|_2 \leq 2\sigma_n \quad \forall s \in \partial \mathbb{E}_\xi \quad \forall j \in \{r+1, \dots, n-1\},$$

8 we repeat the above arguments with $(A, B, -C)$ and $(A_{11}, B_1, -C_1)$ replaced by

$$(A_{11}, B_1, -C_1) \quad \text{and} \quad ((A_{11})_{11}, (B_1)_1, (-C_1)_1),$$

9 respectively. As such, we see \mathbf{H}_{j-1} as the one-step truncation of \mathbf{H}_j . Note that
10 by (A.6) and (A.7), $(A_{11}, B_1, -C_1)$ satisfy the inequalities

$$A_{11} \Sigma_1 + \Sigma_1 A_{11}^* + B_1 B_1^* \leq -2\text{Re}(\xi) A_{11} \Sigma_1 A_{11}^*,$$

11 and

$$A_{11}^* \Sigma_1 + \Sigma_1 A_{11} + C_1^* C_1 \leq -2\text{Re}(\xi) A_{11}^* \Sigma_1 A_{11},$$

12 which are of the form (A.1) and (A.2), respectively.

We now use a telescoping series and the triangle inequality to show that

$$\begin{aligned} \|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2 &= \left\| \sum_{j=r+1}^n [\mathbf{H}_j(s) - \mathbf{H}_{j-1}(s)] \right\|_2 \leq \sum_{j=r+1}^n \|\mathbf{H}_j(s) - \mathbf{H}_{j-1}(s)\|_2 \\ &\leq 2 \sum_{j=r+1}^n \sigma_j \quad \forall s \in \partial \mathbb{E}_\xi, \end{aligned}$$

1 which is (A.4), as required.

2 The proof of (A.5) in the case that $\xi \in i\mathbb{R}$ follows via the same argument used
 3 in [10], the only difference being that the Lyapunov equations (2.2) are replaced by
 4 Lyapunov inequalities (A.1) and (A.2). \square

5 *Proof of Theorem 2.3.* Since (A, B, C, D) is a minimal, balanced and stable, it fol-
 6 lows from [40, Theorem 3.2] that A_{22} is Hurwitz, yielding that $(A_\xi, B_\xi, C_\xi, D_\xi)$
 7 is well-defined for all $\xi \in \mathbb{C}_0 \cup i\mathbb{R}$. Suppose first that $\xi \in \mathbb{C}_0$. Straightforward
 8 algebraic manipulation using the definition of $(A_\xi, B_\xi, C_\xi, D_\xi)$ in (2.5), the decom-
 9 position (2.6) and the equations (2.2) shows that the following Lyapunov inequalities
 10

$$A_\xi \Sigma_1 + \Sigma_1 A_\xi^* + B_\xi B_\xi^* = -2\operatorname{Re}(\xi) A_{12} (\xi I - A_{22})^{-1} \Sigma_2 (\xi I - A_{22})^{-*} A_{12}^* \leq 0, \quad (\text{A.16})$$

11 and

$$A_\xi^* \Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi = -2\operatorname{Re}(\xi) A_{21}^* (\xi I - A_{22})^{-*} \Sigma_2 (\xi I - A_{22})^{-1} A_{21} \leq 0. \quad (\text{A.17})$$

12 hold. If $\xi \in i\mathbb{R}$, then it follows immediately from inspection of (A.16) and (A.17)
 13 that (A_ξ, B_ξ, C_ξ) is balanced, proving statement (ii).

14 We prove statement (i) first assuming that $\xi \in \mathbb{C}_0$. Inequality (A.17) implies that
 15 every eigenvalue of A_ξ has non-positive real part. Suppose that $A_\xi v = \eta i v$ for some
 16 $\eta \in \mathbb{R}$ and $v \in \mathbb{C}^r$. Forming the inner product

$$\langle (A_\xi^* \Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi) v, v \rangle,$$

17 and using (A.17), it follows that

$$0 \leq \|C_\xi v\|^2 = -2\operatorname{Re}(\xi) \langle \Sigma_2 (\xi I - A_{22})^{-1} A_{21} v, (\xi I - A_{22})^{-1} A_{21} v \rangle \leq 0,$$

18 whence

$$\langle \Sigma_2 (\xi I - A_{22})^{-1} A_{21} v, (\xi I - A_{22})^{-1} A_{21} v \rangle = 0,$$

19 as $\operatorname{Re}(\xi) > 0$. Since $\Sigma_2 > 0$, we infer that

$$(\xi I - A_{22})^{-1} A_{21} v = 0.$$

20 Consequently

$$A \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \begin{pmatrix} A_\xi v \\ \xi (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \eta i \begin{pmatrix} v \\ 0 \end{pmatrix},$$

21 and, as A is Hurwitz, we deduce that $v = 0$. Recalling our supposition that $A_\xi v =$
 22 $\eta i v$, we conclude that A_ξ is Hurwitz as well.

For observability, let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be such that $A_\xi v = \lambda v$ and $C_\xi v = 0$. Note that

$$\begin{aligned} A \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} \\ &= \begin{pmatrix} A_\xi v \\ \xi (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix}, \end{aligned}$$

23 so that

$$Az = Ez,$$

24 where

$$E := \begin{pmatrix} \lambda & 0 \\ 0 & \xi \end{pmatrix} \quad \text{and} \quad z := \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix}.$$

1 We conclude that

$$\|e^{At}z\|^2 = \|e^{Et}z\|^2 = \left\| \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\xi t} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|^2 < \|z\|^2 \quad \forall t > 0,$$

2 by [40, Theorem 3.1] applied to the balanced realisation (A, B, C) , so that

$$e^{2\operatorname{Re}(\lambda)t}\|z_1\|^2 + e^{2\operatorname{Re}(\xi)t}\|z_2\|^2 < \|z_1\|^2 + \|z_2\|^2 \quad \forall t > 0.$$

3 Since $\xi \in \mathbb{C}_0$, it follows that

$$z_2 = (\xi I - A_{22})^{-1}A_{21}v = 0,$$

4 from which

$$\lambda v = A_\xi v = A_{11}v + A_{12}(\xi I - A_{22})^{-1}A_{21}v = A_{11}v$$

5 and

$$0 = C_\xi v = C_1v + C_2(\xi I - A_{22})^{-1}A_{21}v = C_1v.$$

6 The pair (C_1, A_{11}) is observable, and so we deduce that $v = 0$, proving that (C_ξ, A_ξ)
7 is observable. The proof that (A_ξ, B_ξ) is controllable is similar, using instead that
8 (A_{11}, B_1) is controllable, and so is omitted.

9 We now consider the situation wherein $\xi \in i\mathbb{R}$. Statement (1) of Lemma 6.2 yields
10 that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is minimal and it is easily shown that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ satisfies the Lyapunov
11 inequalities

$$\mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = -2\operatorname{Re}(\xi)\mathcal{A}\Sigma\mathcal{A}^* \leq 0, \quad (\text{A.18})$$

12 and

$$\mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \mathcal{C}^*\mathcal{C} = -2\operatorname{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{A} \leq 0. \quad (\text{A.19})$$

13 Since $\operatorname{Re}(\xi) = 0$, these simplify to the Lyapunov equations

$$\mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0 \quad \text{and} \quad \mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \mathcal{C}^*\mathcal{C} = 0. \quad (\text{A.20})$$

14 Note that (A.20) implies that \mathcal{A} is Hurwitz and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is balanced. From usual
15 balanced truncation theory [40, Theorem 3.2, Corollary 2], we see that \mathcal{A}_{11} is Hur-
16 witz and $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1)$ is minimal. In particular, it is here where we have used that
17 the singular values are simple, implying that the spectra of Σ_1 and Σ_2 are disjoint.

18 Next, by statement (2) of Lemma 6.2, $\xi \notin \sigma(A_\xi)$, as A is Hurwitz and the equal-
19 ities in (6.6) hold. From these and the minimality of $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1)$ it follows that
20 (A_ξ, B_ξ, C_ξ) is minimal. The Lyapunov equation (A.17) now shows that A_ξ is Hur-
21 witz. \square

22 *Proof of Theorem 2.4:* Let (A, B, C, D) denote a minimal, balanced, stable, reali-
23 sation of \mathbf{G} which, by Theorem 2.3, implies that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well-defined
24 for all $\xi \in \mathbb{C}_0 \cup i\mathbb{R}$. Further, A_ξ is Hurwitz. Let \mathbf{G}_r^ξ , \mathbf{H} and \mathbf{H}_r be defined as in
25 Lemma 6.3. With these choices, we first assume that $\xi \in \mathbb{C}_0$.

26 Invoking statement (3) of Lemma 6.2 to \mathcal{A} and the first equality in (6.6) implies
27 that

$$\sigma(\mathcal{A}), \sigma(\mathcal{A}_{11}) \subseteq \mathbb{E}_\xi. \quad (\text{A.21})$$

28 The error bound (2.9) now follows from subtracting (6.11) from (6.10) in Lemma 6.3
29 and an application of Lemma A.1. In the former result we are using that the map

$$i\mathbb{R} \cup \{\infty\} \ni z \mapsto \frac{1}{z - w},$$

30 a bijection onto $\partial\mathbb{E}_\xi$, where \mathbb{E}_ξ is given by (6.7) and we see from (A.21) that \mathbf{H} and
31 \mathbf{H}_r are well-defined on $\partial\mathbb{E}_\xi$, respectively. In the latter result we take (A, B, C, D)

1 equal to $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Note that the equalities in (A.18) and (A.19) imply that the
 2 inequalities (A.1) and (A.2) respectively hold. That assumption (A.3) holds follows
 3 from (6.6), as every partition in (2.6) gives rise to a Hurwitz A_ξ , by Theorem 2.3.
 4 If $\xi \in i\mathbb{R}$, then the result follows from the error bound (A.5), also in Lemma A.1.
 5 Here we have applied statement (3) of Lemma 6.2 to the first equality in (6.6) to
 6 infer that \mathcal{A}_{11} is Hurwitz. \square

7 **Appendix B. Derivation of (6.24).** Considering (6.17) block wise, we have that

$$\mathcal{A}_{11}^* \Sigma_1 + \Sigma_1 \mathcal{A}_{11} + \mathcal{C}_1^* \mathcal{C}_1 = -\mathcal{K}_1^* \mathcal{K}_1 - q^2 (\mathcal{A}_{11}^* \Sigma_1 \mathcal{A}_{11} + \mathcal{A}_{21}^* \Sigma_2 \mathcal{A}_{21}), \quad (\text{B.1})$$

8 and

$$\Sigma_1 \mathcal{B}_1 - \mathcal{C}_1^* \mathcal{D} = \mathcal{K}_1^* \mathcal{W} - q^2 (\mathcal{A}_{11}^* \Sigma_1 \mathcal{B}_1 + \mathcal{A}_{21}^* \Sigma_2 \mathcal{B}_2) \quad (\text{B.2})$$

9 Given $p \in \partial \mathbb{E}_\xi$, for notational convenience set $\Gamma := (pI - \mathcal{A}_{11})$ and let

$$\mathcal{I}_1 = \mathcal{A}_{11}^* \Sigma_1 \mathcal{A}_{11} + \mathcal{A}_{21}^* \Sigma_2 \mathcal{A}_{21}, \quad \mathcal{I}_2 := \mathcal{A}_{11}^* \Sigma_1 \mathcal{B}_1 + \mathcal{A}_{21}^* \Sigma_2 \mathcal{B}_2.$$

Using (B.1) and (B.2), we compute that

$$\begin{aligned} I - [\mathbf{H}_r^\xi(p)]^* \mathbf{H}_r^\xi(p) &= I - (\mathcal{D} - \mathcal{C}_1(pI - \mathcal{A}_{11})^{-1} \mathcal{B}_1)^* (\mathcal{D} - \mathcal{C}_1(pI - \mathcal{A}_{11})^{-1} \mathcal{B}_1) \\ &= I - (\mathcal{D} - \mathcal{C}_1 \Gamma^{-1} \mathcal{B}_1)^* (\mathcal{D} - \mathcal{C}_1 \Gamma^{-1} \mathcal{B}_1) \\ &= I - \mathcal{D}^* \mathcal{D} + \mathcal{B}_1^* \Gamma^{-*} \mathcal{C}_1^* \mathcal{D} + \mathcal{D}^* \mathcal{C}_1 \Gamma^{-1} \mathcal{B}_1 - \mathcal{B}_1^* \Gamma^{-*} \mathcal{C}_1^* \mathcal{C}_1 \Gamma^{-1} \mathcal{B}_1 \\ &= \mathcal{W}^* \mathcal{W} + q^2 \mathcal{B}_1^* \Sigma_1 \mathcal{B}_1 + q^2 \mathcal{B}_2^* \Sigma_2 \mathcal{B}_2 \\ &\quad + \mathcal{B}_1^* \Gamma^{-*} (\Sigma_1 \mathcal{B}_1 - \mathcal{K}_1^* \mathcal{W} + q^2 \mathcal{I}_2) \\ &\quad + (\mathcal{B}_1^* \Sigma_1 - \mathcal{W}^* \mathcal{K}_1 + q^2 \mathcal{I}_2^*) \Gamma^{-1} \mathcal{B}_1 \\ &\quad + \mathcal{B}_1^* \Gamma^{-*} (\mathcal{A}_{11}^* \Sigma_1 + \Sigma_1 \mathcal{A}_{11} + \mathcal{K}_1^* \mathcal{K}_1 + q^2 \mathcal{I}_1) \Gamma^{-1} \mathcal{B}_1 \\ &= (\mathcal{W} - \mathcal{K}_1 \Gamma^{-1} \mathcal{B}_1)^* (\mathcal{W} - \mathcal{K}_1 \Gamma^{-1} \mathcal{B}_1) + \mathcal{R}, \end{aligned} \quad (\text{B.3})$$

where

$$\begin{aligned} \mathcal{R} &:= q^2 \mathcal{B}_2^* \Sigma_2 \mathcal{B}_2 + q^2 \mathcal{B}_1^* \Gamma^{-*} \mathcal{I}_2 + q^2 \mathcal{I}_2^* \Gamma^{-1} \mathcal{B}_1 \\ &\quad + \mathcal{B}_1^* \Gamma^{-*} (q^2 \Gamma^* \Sigma_1 \Gamma + \Sigma_1 \Gamma + \Gamma^* \Sigma_1 + \mathcal{A}_{11}^* \Sigma_1 + \Sigma_1 \mathcal{A}_{11} + q^2 \mathcal{I}_1) \Gamma^{-1} \mathcal{B}_1 \\ &= q^2 [\mathcal{B}_2^* \Sigma_2 \mathcal{B}_2 + \mathcal{B}_1^* \Gamma^{-*} (\mathcal{A}_{11}^* \Sigma_1 \mathcal{B}_1 + \mathcal{A}_{21}^* \Sigma_2 \mathcal{B}_2) \\ &\quad + (\mathcal{B}_1^* \Sigma_1 \mathcal{A}_{11} + \mathcal{B}_2^* \Sigma_2 \mathcal{A}_{21}) \Gamma^{-1} \mathcal{B}_1] \\ &\quad + \mathcal{B}_1^* \Gamma^{-*} (q^2 \Gamma^* \Sigma_1 \Gamma + 2\text{Re}(p) \Sigma_1 + q^2 (\mathcal{A}_{11}^* \Sigma_1 \mathcal{A}_{11} + \mathcal{A}_{21}^* \Sigma_2 \mathcal{A}_{21})) \Gamma^{-1} \mathcal{B}_1 \\ &= q^2 (\mathcal{B}_2 + \mathcal{A}_{21} \Gamma^{-1} \mathcal{B}_1)^* \Sigma_2 (\mathcal{B}_2 + \mathcal{A}_{21} \Gamma \mathcal{B}_1) \\ &\quad + \mathcal{B}_1^* \Gamma^{-*} (q^2 (\Gamma^* \Sigma_1 \Gamma + \mathcal{A}_{11}^* \Sigma_1 \mathcal{A}_{11} + \Gamma^* \Sigma_1 \mathcal{A}_{11} + \mathcal{A}_{11}^* \Sigma_1 \Gamma) + 2\text{Re}(p) \Sigma_1) \Gamma^{-1} \mathcal{B}_1 \\ &= q^2 (\mathcal{B}_2 + \mathcal{A}_{21} \Gamma^{-1} \mathcal{B}_1)^* \Sigma_2 (\mathcal{B}_2 + \mathcal{A}_{21} \Gamma \mathcal{B}_1) \\ &\quad + 2 \mathcal{B}_1^* \Gamma^{-*} (\text{Re}(p) + \text{Re}(\xi) |p|^2) \Sigma_1 \Gamma^{-1} \mathcal{B}_1 \\ &= q^2 (\mathcal{B}_2 + \mathcal{A}_{21} \Gamma^{-1} \mathcal{B}_1)^* \Sigma_2 (\mathcal{B}_2 + \mathcal{A}_{21} \Gamma \mathcal{B}_1). \end{aligned} \quad (\text{B.4})$$

10 In the final equality above we have used that $p \in \partial \mathbb{E}_\xi$ and (6.9). Combining (B.3)
 11 and (B.4) gives (6.24), as required.

Appendix C. Derivation of (6.29). The arguments are identical in spirit to those used in Appendix B. Given $p \in \partial\mathbb{E}_\xi$, for notational convenience set $\Theta := (pI - \mathcal{A})$. Using (6.17), we compute that

$$\begin{aligned}
I - [\mathbf{H}(p)]^* \mathbf{H}(p) &= I - (\mathcal{D} - \mathcal{C}(pI - \mathcal{A})^{-1} \mathcal{B})^* (\mathcal{D} - \mathcal{C}(pI - \mathcal{A})^{-1} \mathcal{B}) \\
&= I - (\mathcal{D} - \mathcal{C}\Theta^{-1} \mathcal{B})^* (\mathcal{D} - \mathcal{C}\Theta^{-1} \mathcal{B}) \\
&= I - \mathcal{D}^* \mathcal{D} + \mathcal{B}^* \Theta^{-*} \mathcal{C}^* \mathcal{D} + \mathcal{D}^* \mathcal{C} \Theta^{-1} \mathcal{B} - \mathcal{B}^* \Theta^{-*} \mathcal{C}^* \mathcal{C} \Theta^{-1} \mathcal{B} \\
&= \mathcal{W}^* \mathcal{W} + q^2 \mathcal{B}^* \Sigma \mathcal{B} + \mathcal{B}^* \Theta^{-*} (\Sigma \mathcal{B} - \mathcal{K}^* \mathcal{W} + q^2 \mathcal{A}^* \Sigma \mathcal{B}) \\
&\quad + (\mathcal{B}^* \Sigma - \mathcal{W}^* \mathcal{K} + q^2 \mathcal{B}^* \Sigma \mathcal{A}) \Theta^{-1} \mathcal{B} \\
&\quad + \mathcal{B}^* \Theta^{-*} (\mathcal{A}^* \Sigma + \Sigma \mathcal{A} + \mathcal{K}^* \mathcal{K} + q^2 \mathcal{A}^* \Sigma \mathcal{A}) \Theta^{-1} \mathcal{B} \\
&= (\mathcal{W} - \mathcal{K} \Theta^{-1} \mathcal{B})^* (\mathcal{W} - \mathcal{K} \Theta^{-1} \mathcal{B}) + \mathcal{S}. \tag{C.1}
\end{aligned}$$

Here

$$\begin{aligned}
\mathcal{S} &:= \mathcal{B}^* \Theta^{-*} (q^2 (\Theta^* \Sigma \Theta + \mathcal{A}^* \Sigma \mathcal{A} + \mathcal{A}^* \Sigma \Theta + \Theta^* \Sigma \mathcal{A}) + 2\text{Re}(p)\Sigma) \Theta^{-1} \mathcal{B} \\
&= 2\mathcal{B}^* \Theta^{-*} (\text{Re}(p) + \text{Re}(\xi)|p|^2) \Sigma \Theta^{-1} \mathcal{B} \\
&= 0. \tag{C.2}
\end{aligned}$$

- 1 In the final equality above we have used that $p \in \partial\mathbb{E}_\xi$ and (6.9). Combining (C.1)
2 and (C.2) gives (6.29), as required.

- 3 **Appendix D. Proof of Lemma 6.5.** The proof is by direct calculation. For
4 notation convenience, set $\Psi := (\xi I - A_{22})^{-1}$, $\Phi := (I + D)^{-1}$ and

$$X_B := B_2 \Phi, \quad X_C := C_2 \Psi, \quad N := (I + X_C X_B)^{-1}, \quad M := (I + X_B X_C)^{-1}. \tag{D.1}$$

- 5 Note that M and N are well-defined by our assumption that all the terms which
6 appear in the commuting diagram are. Straightforward calculations show that

$$N = I - X_C X_B N, \quad X_B N = M X_B, \quad \text{and} \quad X_C M = N X_C. \tag{D.2}$$

Using the definitions in (2.5), (6.39) and (D.1) and the properties (D.2), we have that

$$\begin{aligned}
\widetilde{(A_\xi)} &= A_\xi - B_\xi (I + D_\xi)^{-1} C_\xi \\
&= A_\xi - (B_1 + A_{12} \Psi B_2) (I + D + C_2 \Psi B_2)^{-1} (C_1 + C_2 \Psi A_{21}) \\
&= A_\xi - (B_1 \Phi + A_{12} \Psi B_2 \Phi) (I + C_2 \Psi B_2 \Phi)^{-1} (C_1 + C_2 \Psi A_{21}) \\
&= A_\xi - (B_1 \Phi + A_{12} \Psi X_B) N (C_1 + X_C A_{21}) \\
&= A_\xi - (B_1 \Phi + A_{12} \Psi X_B) (I - X_C X_B N) (C_1 + X_C A_{21}). \tag{D.3}
\end{aligned}$$

Similarly

$$\begin{aligned}
(\tilde{A})_\xi &= (\tilde{A})_{11} + (\tilde{A})_{12} (\xi I - (\tilde{A})_{22}) (\tilde{A})_{21} \\
&= (A - B \Phi C)_{11} + (A - B \Phi C)_{12} (\xi I - (A - B \Phi C)_{22})^{-1} (A - B \Phi C)_{21} \\
&= A_{11} - B_1 \Phi C_1 + (A_{12} - B_1 \Phi C_2) (\xi I - A_{22} + B_2 \Phi C_2)^{-1} (A_{21} - B_2 \Phi C_1) \\
&= A_{11} - B_1 \Phi C_1 + (A_{12} \Psi - B_1 \Phi C_2 \Psi) (I + B_2 \Phi C_2 \Psi)^{-1} (A_{21} - B_2 \Phi C_1) \\
&= A_{11} - B_1 \Phi C_1 + (A_{12} \Psi - B_1 \Phi X_C) M (A_{21} - X_B C_1). \tag{D.4}
\end{aligned}$$

Inspection of (D.3) and (D.4) reveals that they are equal. Next, we compute that

$$\begin{aligned}
\frac{1}{\sqrt{2}}(\widetilde{B}_\xi) &= B_\xi(I + D_\xi)^{-1} = (B_1 + A_{12}\Psi B_2)(I + D + C_2\Psi B_2)^{-1} \\
&= (B_1\Phi + A_{12}\Psi B_2\Phi)(I + C_2\Psi B_2\Phi)^{-1} = (B_1\Phi + A_{12}\Psi X_B)N \\
&= B_1\Phi + (A_{12}\Psi - B_1\Phi X_C)MX_B \\
&= B_1\Phi + (A_{12}\Psi - B_1\Phi C_2\Psi)(I + B_2\Phi C_2\Psi)^{-1}X_B \\
&= B_1\Phi + (A_{12} - B_1\Phi C_2)(\xi I - A_{22} + B_2\Phi C_2)^{-1}B_2\Phi \\
&= \frac{1}{\sqrt{2}}((\tilde{B})_1 + (\tilde{A})_{12}(\xi I - (\tilde{A})_{22})^{-1}(\tilde{B})_2) = \frac{1}{\sqrt{2}}(\tilde{B})_\xi.
\end{aligned}$$

Further,

$$\begin{aligned}
-\frac{1}{\sqrt{2}}(\widetilde{C}_\xi) &= (I + D_\xi)^{-1}C_\xi = (I + D + C_2\Psi B_2)^{-1}(C_1 + C_2\Psi A_{21}) \\
&= \Phi(I + C_2\Psi B_2\Phi)^{-1}(C_1 + C_2\Psi A_{21}) = \Phi N(C_1 + X_C A_{21}) \\
&= \Phi C_1 + \Phi X_C M(A_{21} - X_B C_1) \\
&= \Phi C_1 + \Phi C_2\Psi(I + B_2\Phi C_2\Psi)^{-1}(A_{21} - B_2\Phi C_1) \\
&= \Phi C_1 + \Phi C_2(\xi I - A_{22} + B_2\Phi C_2)^{-1}(A_{21} - B_2\Phi C_1) \\
&= -\frac{1}{\sqrt{2}}((\tilde{C})_1 + (\tilde{C})_2(\xi I - (\tilde{A})_{22})^{-1}(\tilde{A})_{21}) = -\frac{1}{\sqrt{2}}(\tilde{C})_\xi.
\end{aligned}$$

Finally,

$$\begin{aligned}
(\widetilde{D}_\xi) &= (I - D_\xi)(I + D_\xi)^{-1} = (I - D - C_2\Psi B_2)(I + D + C_2\Psi B_2)^{-1} \\
&= ((I - D)\Phi - C_2\Psi B_2\Phi)(I + C_2\Psi B_2\Phi)^{-1} = (\tilde{D} - X_C X_B)N \\
&= \tilde{D} - 2\Phi X_C M X_B \\
&= \tilde{D} - 2\Phi C_2\Psi(I + B_2\Phi C_2\Psi)^{-1}B_2\Phi \\
&= \tilde{D} - 2\Phi C_2(\xi I - A_{22} + B_2\Phi C_2)^{-1}B_2\Phi = \tilde{D} + (\tilde{C})_2(\xi I - (\tilde{A})_{22})^{-1}(\tilde{B})_2 \\
&= (\tilde{D})_\xi.
\end{aligned} \tag{D.5}$$

1 To establish (D.5) we used that

$$\tilde{D} - \tilde{D}N + X_C X_B N - 2\Phi X_C M X_B = 0.$$

2 The proof is complete. \square

3

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